From Hough Transform to Integral Geometry

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Abstract— **Hough Transform (H.T.) is a classical tool for multiple alignment detection in image processing, based on the property that Aligned Points are transformed into Intersecting Curves (APIC). Among the alternative transforms which possess the APIC property, one of the most interesting is Polar Transform (P.T.) which exchanges a point (a,b) and the straight line with equation ax + by = 1. This transform represents classical duality between a pole and its polar line w.r. to the unit circle. Another property common to both H.T. and P.T. is the correspondence between shapes' Boundary Length and connected Areas (BLA property), either direct or compensated by a weight function, allowing an efficient measure of these lengthes on a digital screen. P.T. is shown to be connected to Projective and Integral (or Stochastic) Geometry with an important role given to a weight function $1/d^3$.**

I. Introduction

This paper is motivated by a long practise in image processing especially in the domain of line detection and length calculation. Instead of using "ad hoc" ameliorations of Hough Transform (H.T.), we have developed alternative transforms, [1],[2], in connection with other geometrical tools. One of them is Polar Transform (P.T.), a version of point/line duality in the plane, is especially important (P.T. should not be confused with the so-named "straight line Hough Transform"). This paper shows similarity aspects between H.T. and P.T., and the connection of P.T. with projective and integral (or stochastic) geometry.

In a first part, a family of measures on sets of lines is introduced. Then, different definitions of H.T. for points and shapes are given. In a third part, P.T. is introduced. Both transforms are shown to possess similar "APIC" and "BLA" properties (to be defined later).

II. Measures on sets of lines

Let L_P be the set of lines intersecting a set of points P in the plane (usually, P is a convex "shape"). For example, L_{ABC} is the set of straight lines which intersect a triangle ABC, $L_{\{A\}}$ is the pencil of lines passing through point A.

Let us denote $L_{A,BC} = L_{(AB)} \cap L_{(AC)}$ which is the set of lines separating A from B and C. We have

$L_{ABC} = L_{BC,A} \cup L_{CA,B} \cup L_{AB,C} \cup P$ (1)

(disjoint union, with $P = L_{\{A\}} \cup L_{\{B\}} \cup L_{\{C\}}$).

Such set can be given a measure such that $\mu(L_{\{A\}}) = 0$ for pencils of lines. As a consequence of (1):

$\mu(L_{ABC}) = \mu(L_{BC,A}) + \mu(L_{CA,B}) + \mu(L_{AB,C})$ (2)

If measure μ is such that : $\mu(L_{BC,A}) = b + c - a$ (3)

(2) becomes :

$\mu(L_{BC,A}) = (b + c - a) + (c + a - b) + (a + b - c)$

i.e., $\mu(L_{ABC}) = 2(a + b + c)$ (4)

Thus, *the measure of a set of lines which intersects a triangular shape is the (double of the) length of its boundary.* This property has a wider application range: it remains true for all convex polygons, and then, by continuity arguments, to almost any *convex* shape S. In the case where $\mu(S)$ is connected to the area (or compensated area) of S, one will say that it has the Boundary Length vs. Area (BLA) property.

Remark : (3) is fundamental because it expresses the "excess" in triangular equality: $b + c \geq a$.

III. Classical H.T.

A straightforward definition of H.T. is: to a point P_0 with polar coordinates (θ_0, p_0) in the original plane, is associated the curve with equation

$p = p_0 cos(\theta - \theta_0)$ (5)

This sine curve is drawn in a special "representation strip" $RS = [0, 2\pi) \times (-\infty, +\infty)$ with (θ, p) considered as rectangular (cartesian) coordinates (Fig. 1a). This definition permits a straightforward proof of the correspondence between Aligned Points and Intersecting Curves (APIC property).

But the definition above has to be enlarged as follows for a better understanding. Let $\Delta_{\theta,p}$ be the straight line with equation $x\cos(\theta) + y\sin(\theta) - p = 0$. We consider $\Delta_{\theta,p}$ as a *point* in RS. Let us consider a fixed point $P_0\,(x_0, y_0) = (p_0\cos(\theta_0), p_0\sin(\theta_0))$; A line $\Delta_{\theta,p}$ contains this point if and only if $p_0\cos(\theta_0)\cos(\theta) + p_0\sin(\theta_0)\sin(\theta) - p = 0$: we are back to definition (5). In other words, the set of points of the curve with equation (5) represents the set of lines belonging to the pencil $L_{\{P_0\}}$.

Remark : H.T. is redundant because $\Delta_{\theta+\pi,p} = \Delta_{\theta,-p}$.

In a natural way, the H.T. of a "shape" S (notation HT(S)) is the set of straight lines $D_{\theta,p}$ intersecting S.

The measure $\mu(S)$ is, in a plain manner, the area of set HT(S). It is easy to show that this area is displacement-invariant (Fig 1, 2). The gray-colored set of points represents the set of lines intersecting *both* line segment a_1b_1 and line segment c_1d_1 (note the natural redundancy: the two gray "quadrilaterals" represent the same set of straight lines).

We are now able to explain what is the Boundary Length vs. Area (BLA) property. Indeed, property (4) above is easy to establish. Consider Fig. 3. The area representing the set of lines L_{ABC} in the strip RS is obtained by integration of the "shadows" of the projection of line segments AB,

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BC and CA on a turning axis: each segment of length L contributes twice, by an amount $2L\int_0^{2\pi}|\cos(\theta)|\,d\theta = 4L$; in this way, we get twice the looked for area, $2\mu(S) = 4(a+b+c)$ (see (4)).

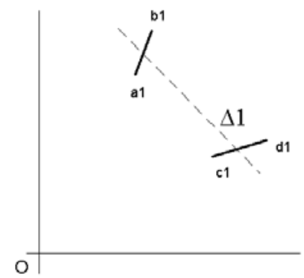

Fig. 1.

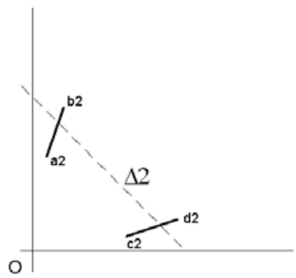

Fig. 2.

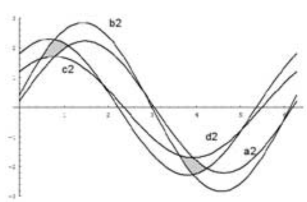

Fig. 3.

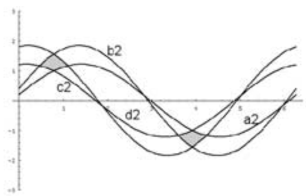

Fig. 4.

IV. Polar Transform (P.T.)

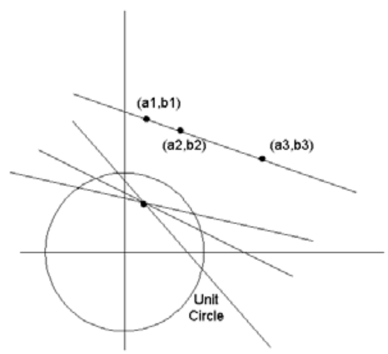

Fig. 5.

By definition, P.T. exchanges point (a,b), called a *pole*, and straight line with equation ax + by = 1, called the *polar line* associated to the given pole, and vice versa. Let S be a certain set of lines; let us denote by $SOP(S)$ the Set Of Poles of the lines of S.

It is easy to check that if all the lines of S pass through a same point A (Fig. 5), $SOP(S)$ has all its points aligned (on a line which is the polar line of A): thus, P.T. verifies the APIC property.

A particular case (Fig. 5): if Q is a (convex) quadrilateral, $SOP(L_Q)$ is either a quadrilateral in the general case or is made of two unbounded polygonal components if certain lines of S pass through the origin.

For a general shape P, the area of $SOP(L_P)$ depends on the location of P in the plane (Fig. 7): the farthest from the origin, the smallest the area. In an unexpected way, there is a simple compensating weight function $1/D^3$ as we are going to prove it: in this way, we will have obtained the BLA correspondence for P.T. as we had it for H.T.

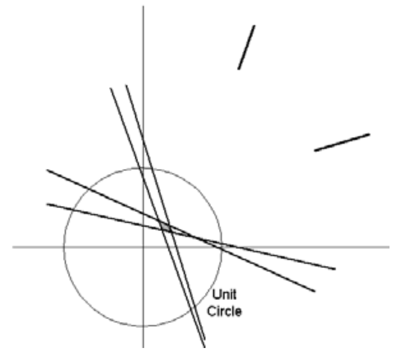

Fig. 6.

In order to obtain this $1/D^3$ weight, we will use projective geometry. Let us recall that, being given a 3×3 matrix $M=(m_{ij})$, the projective function $(x',y') = P_M(x,y)$ associated to M is defined by:

$$x' = \frac{N_1}{D}\ ,\ y' = \frac{N_2}{D} \text{ with } \begin{bmatrix} N_1 \\ N_2 \\ D \end{bmatrix} = \begin{bmatrix} m_{11}x+m_{12}y+m_{13} \\ m_{21}x+m_{22}y+m_{23} \\ m_{31}x+m_{32}y+m_{33} \end{bmatrix}$$

Lemma : $\det\begin{bmatrix} \partial x'/\partial x & \partial y'/\partial x \\ \partial x'/\partial y & \partial y'/\partial y \end{bmatrix} = det(M)/D^3.$

Proof left to the reader. Recall here the geometrical meaning of this jacobian: it is equal to the ratio of elementary areas.

Let us now consider two "small" line segments (see Fig. 7): $D_1+ds_1\overrightarrow{V_1}$ and $D_2+ds_2\overrightarrow{V_2}$ with points D_k (x_k,y_k), unitary vectors $V_k=(a_k,b_k)$ and $\mu_k = ds_k$ $(k=1,2)$. Let S be the set of straight lines intersecting both line segment, whose poles have the following coordinates:

$$\begin{aligned} a &= \tfrac{1}{\delta}\left[(y_1+\mu_1 b_1)-(y_2+\mu_2 b_2)\right] \\ b &= \tfrac{1}{\delta}\left[(x_2+\mu_2 a_2)-(x_1+\mu_1 a_1)\right] \end{aligned}$$

$$\text{where } \delta = \det\begin{bmatrix} x_1+\mu_1 a_1 & x_2+\mu_2 a_2 \\ y_1+\mu_1 b_1 & y_2+\mu_2 b_2 \end{bmatrix}$$

We observe that (a,b) is a projective function of μ_1 and μ_2, because the second order infinitesimal $\mu_1\mu_2$ term disappears. Using now the lemma and a certain factorization, the set $SOP(S)$ of all poles associated to lines which hit both line segments (which is an infinitesimal quadrilateral, as seen before) has the following area:

$$dA = ds_1 ds_2 \frac{\det\begin{bmatrix} a_1 & x_2-x_1 \\ b_1 & y_2-y_1 \end{bmatrix} \det\begin{bmatrix} a_2 & x_1-x_2 \\ b_2 & y_1-y_2 \end{bmatrix}}{\det\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^3}$$

This formula can be written in this way:

$$dA = ds_1 ds_2 \frac{\det\left[\overrightarrow{D_1D_2}, \overrightarrow{V_1}\right] \det\left[\overrightarrow{D_1D_2}, \overrightarrow{V_2}\right]}{\det\left[\overrightarrow{OD_1}, \overrightarrow{OD_2}\right]^3} \quad (6)$$

Integration of (6) gives, for a general set of lines $S = L_P$ intersecting a shape P:

$$area(SOP(L_P)) = \int\int_{\partial P \times \partial P} ds_1 ds_2 \frac{\sin\alpha_1 \sin\alpha_2}{d} = L(\partial P) \quad (7)$$

where $d = d_{s_1,s_2} = \left\| \overrightarrow{D_1D_2} \right\|$, $\alpha_k = angle(\overrightarrow{D_1D_2}, \overrightarrow{V_k})$, and s_k = curvilinear abscissa of D_k $(k = 1, 2)$ on the boundary ∂P of the shape.

The second equality in formula (7) is a classical expression in Integral (Stochastic) Geometry [3] for the length of the boundary ∂P, the "best" displacement-invariant result that could be obtained...

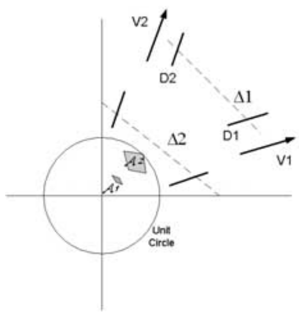

Fig. 7.

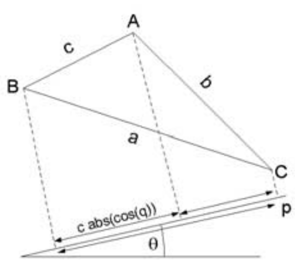

Fig. 8.

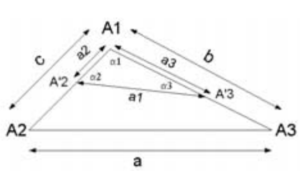

Fig. 9.

Let us apply formula (7). Fig. 9 represents a triangle $T = A_1A_2A_3$ with sides' lengthes a, b, c. Let A'_2 (resp. A'_3) the points where a generic straight line intersects A_1A_2 (resp. A_1A_3). Let $a_1 = A'_2A'_3$, $a_2 = A_1A'_3$, $a_3 = A_1A'_2$. Using classical relationships:

$$\frac{\sin\alpha_1}{a_1} = \frac{\sin\alpha_2}{a_2} = \frac{\sin\alpha_3}{a_3} \; and \; a_1^2 = a_2^2 + a_3^2 - 2a_2a_3\cos\alpha_1$$

formula (7) gives :

$$area\ (SOP(P)) = \int_{a_2=0}^{b}\int_{a_3=0}^{c} \frac{1}{a_1}\sin\alpha_2 \sin\alpha_3 da_2 da_3$$

$$= (\sin a_1)^2 \int_{a_2=0}^{b}\int_{a_3=0}^{c} \frac{a_2a_3}{a_1^3} da_2 da_3$$

$$= (\sin a_1)^2 \int_{a_2=0}^{b}\int_{a_3=0}^{c} \frac{a_2a_3}{(a_2^2 + a_3^2 - 2a_2a_3\cos\alpha_1)^{3/2}} da_2 da_3$$

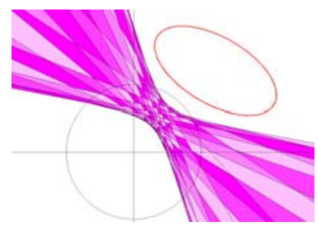

Fig. 10.

yielding: $area\ (SOP(S)) = b + c - a$ (property (3)): we have the same displacement invariant measure. Let us end with a particular application: consider a (convex) polygon with n sides; the set S of lines intersecting the quadrilateral is partitioned (up to zero-measure sets) into $\frac{n(n-1)}{2}$ "tiles", each tile corresponding to a pairing between two sides of the quadrilateral (Fig. 10).

Remark : The "envelope" in Fig. 10 is inscribed into an hyperbola because the initial convex polygon is inscribed into an ellipse; this is one of the interesting properties of P.T.: the images of conic curves are conic curves (seen in a dual way: from a tangentially-defined curve to a point-defined curve).

V. Conclusion

This paper has shown that H.T. and P.T. share similar properties: APIC and BLA, with a $1/d^3$ weight in the case of P.T. The superiority of P.T. lies in the fact that it allows more connections with other domains of mathematics, especially with projective and integral geometry, but also Legendre Transform, etc. Moreover, all these connections can be easily extended to 3D.

References

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