Points Alignment and Curves Intersection in Image Processing: a duality framework

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Abstract

This article introduces duality as a key concept for a geometrical understanding of the Hough Transform (H.T.). Both the H.T., which associates a sine curve to a point, and the Duality Transform (D.T.) which associates a straight line to a point, exchange Aligned Points with Intersecting Curves (A.I. Transforms). The main objective of this paper is to introduce other A.I. transforms, to establish the close connection between these different transforms, and to explain their respective interest.

Key words: Image Processing, Hough Transform, Geometry, Duality, Inversion, Alignment Detection.

AMS subject classifications: 51A50, 51B05, 68U10.

1 Introduction

The Hough Transform (H.T.) is a point-to-curve transform [6] which associates to a point with polar coordinates r_0, θ_0 the sine curve with equation $r = r_0 \cos(\theta - \theta_0)$. It is one of the most useful tools in image processing for the detection of points alignment.

H.T. is an "A.I. transform", this acronym meaning in this paper that any set of Aligned points is transformed into a set of Intersecting curves, in this case sine curves.

These curves have a common point which contains the needed information for the "reconstruction" of the exact straight line on which are situated the initial points. This reconstruction, which maps a point to a line, cannot be assimilated to an inverse transform [4]; the inverse of a transform which maps a point onto a curve should map a curve onto a point. It is why this kind of transform has to be placed in another context, which is provided by duality.

In order to keep things tractable, we will use duality with respect to the unit circle, one of the simplest conic curves.

A fundamental principle is that any transform composition "T_1 followed by T_2", where T_1 is an A.I. transform and T_2 is (almost) any continuous point-to-point transform, is itself an A.I. transform.

This remark will help to build different A.I. transforms using a fundamental transform that will be the Duality Transform. With the help of different representations, their inter-connection with H.T. will be established. Their interest as possible substitutes to H.T. will be discussed.

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2 Inversion and Duality Transform

$R\backslash i$

Inversion is a point-to-point transform in $R^2\backslash\{P\}$ (more generally in $R^n\backslash\{P\}$), denoted by $I_{P,\lambda}$, where P is a point and λ a nonzero real number; we will say that the image of M is $M' = I_{P,\lambda}(M)$ iff

$$(1) \qquad \overrightarrow{PM'} = \lambda \, \overrightarrow{PM} / \left\| \overrightarrow{PM} \right\|^2$$

An equivalent definition uses two conditions: P, M, M' are aligned and $\overrightarrow{PM'}.\overrightarrow{PM} = \lambda$.
This definition is involutory: the image of M' is M.

If P is the origin and $\lambda = 1$, we set $I_{0,1} = I$. The Duality Transform (D.T.) is defined as the association to a point $M(a, b)$ of a straight line in R^2, passing through $M' = I(M)$ and having $\overrightarrow{OM}$ as its normal vector i.e., the straight line with equation $ax + by = 1$ (Fig. 1).

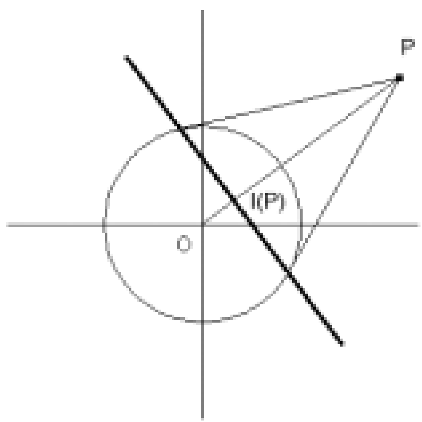

Figure 1: The Duality Transform (D.T.)

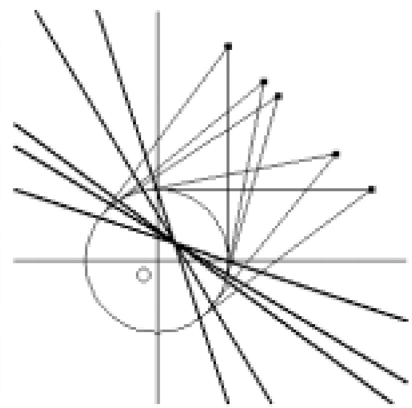

Figure 2: The D.T. is an A.I. transform: a set of aligned points gives a pencil of straight lines.

Let U denote the unit circle. If point M_0 is outside U, one can easily show that the D.T. associates to M_0 the straight line joining the contact points of the two tangent lines to U issued from M_0.
The following result is easy to establish:

Proposition 2.1 *2D (resp. 3D) inversion exchanges a straight line (resp. a plane) not passing through the origin and a circle (resp. a sphere) passing through the origin.*

Proposition 2.2 .

1. *The D.T. associates, to a set of aligned points, a set of intersecting straight lines (a pencil of lines), i.e., The D.T. is an A.I. transform (Fig. 2).*

2. *Inversion $I_{0,2}$ applied to this pencil of lines yields a "pencil of circles" , i.e., a set of circles passing through two common points, the origin O and its symmetrical point O' w.r. to the common line L (Fig. 3).*

Proof It is sufficient to establish this property for 3 points $M_k(x_k, y_k)$ $(k = 1, 2, 3)$ and their 3 "dual" lines, which is easy. ∎

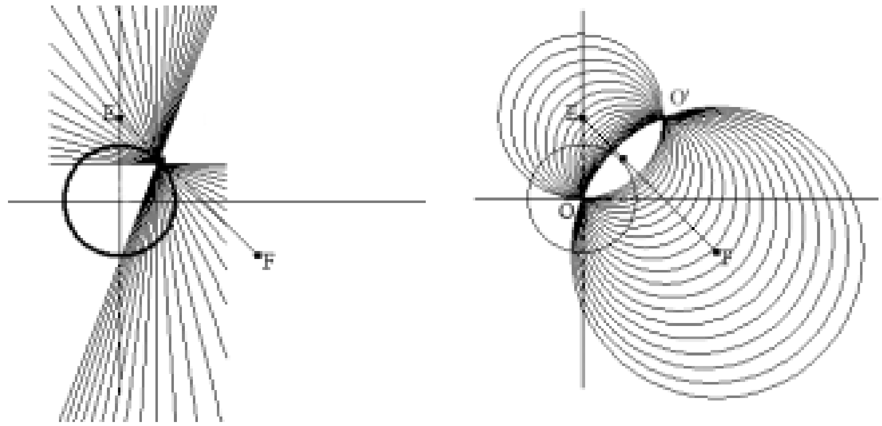

Figure 3: a) and b) The "Circle Transform" CT_0 can be defined in two steps; the D.T. first transforms line segment EF into a pencil of straight lines, which is then transformed by inversion $I_{0,2}$ into a pencil of circles.

3 Duality Transform and Hough Transform

From a practical point of view, the line on which are situated certain aligned points is unknown; the "old" image is replaced by a new one made of the union of the circles centered on these points which pass through the origin. In the case of a perfect alignment, there is, for all these circles, a unique common point O' other than the origin O (Fig. 3b).

A non-perfect alignment gives a "cluster of circles" which, by image processing techniques, yields a "best approximation" O'.

In both cases, the unknown line is recovered as the perpendicular bisector of O and O'.

The way we have associated a point to a circle is essentially the same as the way a point is associated to a sine curve with the H.T. Let us see why.

The cartesian equations of the above mentioned circles associated to points $M_k(x_k, y_k)$ are:

(2) $$x^2 + y^2 - 2x_k x - 2y_k y = 0$$

which, using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and $x_k = r_k\cos(\theta_k)$, $y_k = r_k\sin(\theta_k)$, become the polar equations:

(3) $$\rho = r_k\cos(\theta - \theta_k)$$

It means that, in the "Hough plane" $H = [-\pi; \pi) \times (-\infty; \infty)$, each point M_k can be associated to the sine curve with equation (3). We have regained the "classical" H.T. with a pencil of sine curves instead of a pencil of circles.

4 Other A.I. Transforms

For every fixed c, let us denote by CT_c the "Circle Transform" which maps the point (a, b) onto the circle with equation:

(4) $$x^2 + y^2 - 2ax - 2by + c = 0$$

The following result is easy to establish in the same way as proposition (2.1):

Proposition 4.1 *For any c, CT_c are A.I. transforms.*

Let us restrict our attention to the specific cases CT_{-1} and CT_1, as depicted on Figures (4,5).

The CT_1 (resp. CT_{-1}) associates, to a point M, the circle centered in M which intersects orthogonaly (resp. intersecting along a diameter) the unit circle U. These two transforms are bounded with the following meaning: each point of a circle which is *inside* the unit disk U has an image by inversion I which is outside U.

Remarks:

1. The CT_1 does not attribute an image to the points inside the unit circle U.

2. We have shown [3] that the CT_{-1}, CT_0 and CT_1 provide models for the three fundamental geometries (resp. non-euclidean spherical, euclidean, non-euclidean hyperbolic).

3. The CT_c could have been presented as the composition of D.T. with "generalized inversion" [3] defined as the point-to-point transform $J_{P,\lambda,c}$: $M \to M'$ with $\overrightarrow{PM'} = \lambda\, \overrightarrow{PM}/(\left\|\overrightarrow{PM}\right\|^2 + c)$.

   This is due to the fact that, if $X = (x, y)$ and $M = (a, b)$, then $M^T.J_{0,2,c}(X) = 1$ is equivalent to equation (4).

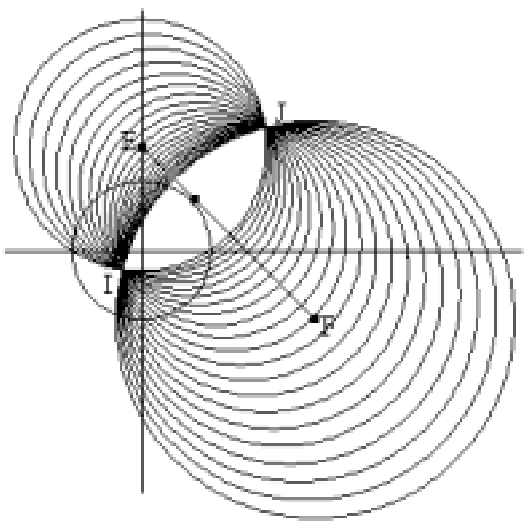

Figure 4: Circle Transform CT_{-1}

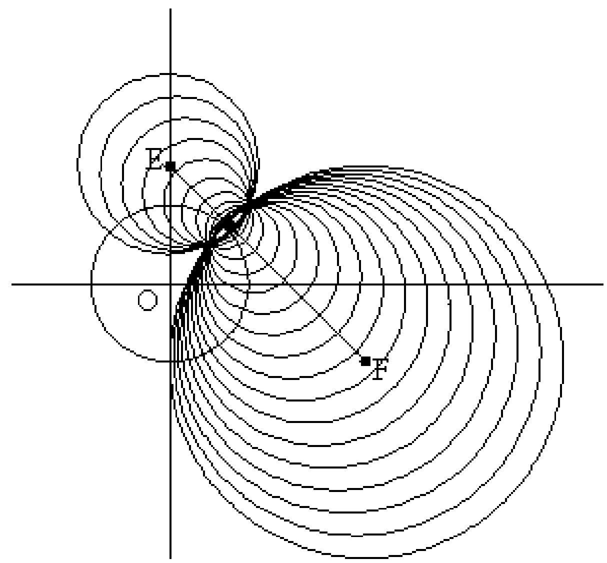

Figure 5: Circle Transform CT_1

5 A unified view

The CT_c can be settled in an interesting 3D context.

Let Σ be the unit sphere and $S(0, 0, -1)$ (resp. $N(0, 0, 1)$) be its South pole (resp. North pole).

To a point $M(a, b)$, we associate the plane Π_M and the sphere Σ_M with respective equations:

(5)
$$\begin{cases} ax + by - z = 0 \\ x^2 + y^2 + z^2 - 2ax - 2by - 1 = 0 \end{cases}$$

3D Inversion $I_{S,2}$ exchanges Π_M and Σ_M (straightforward computation).

If certain points M_k belong to a common line in the original plane, spheres Σ_{M_k} share a common vertical circle passing through S and N; it is a pencil of spheres.

The intersections of spheres Σ_M with level planes: $z = 0, z = 1$ and $z = \sqrt{2}$ give resp. the CT_{-1}, CT_0 and CT_1 (Fig. 6).

A supplementary interest of this 3D representation is that it allows a new connection with H.T. Let C be the vertical cylinder circumscribed to the unit sphere Σ (Fig. 7). The traces of planes π_M on C are ellipses. If C is "unfolded", sine curves appear with equation (3): one recovers the classical H.T.

6 Yet another unification

There is another possible unification provided, once again, by a certain 3D space, this time a parameter space.

Every circle with equation (4) can be considered as a point (a, b, c) in an "abstract" parameter space called the "space of circles" and denoted by SC [1][2]. Parameters (a, b, c) are linked to the radius r of the circle by relationship:

(6)
$$r^2 = a^2 + b^2 - c$$

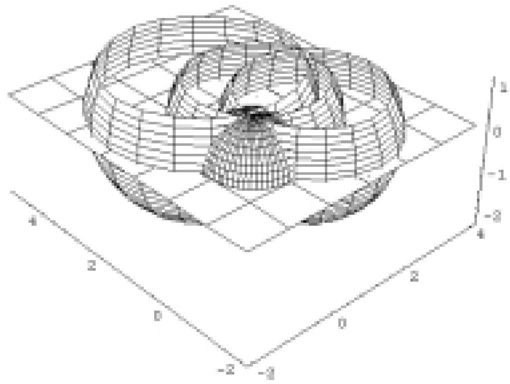

Figure 6: The smallest sphere is unit sphere Σ. The centers of the three "large" spheres are aligned points located in $z = 0$ plane. The effect of Circle Transforms CT_k can be seen as traces on the horizontal plane $z = \sqrt{1+k}$.

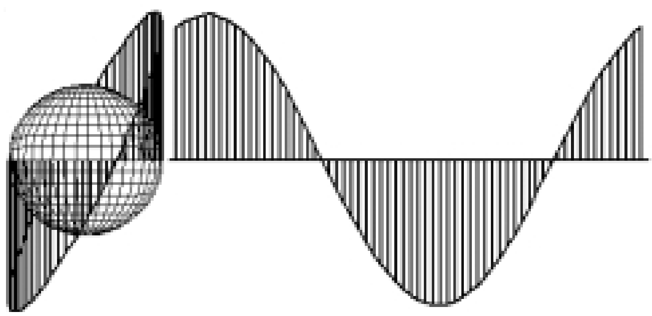

Figure 7: The intersection of a plane with the unit cylinder gives an ellipse, which, unfolded, gives a sine curve.

The set of *point circles (r=0)* is the paraboloid with equation $a^2 + b^2 = c$. Its interior set is a forbidden set. Relationship (6) can be considered as a (projective) quadratic form on SC. The derived inner product is:

$$(\sigma \mid \sigma') = aa' + bb' - (c + c')/2 \qquad (7)$$

Relationship $(\sigma \mid \sigma') = 0$ expresses the usual orthogonality of circles. The set of circles which are orthogonal to a given circle σ is described by a linear equation, i.e., is a plane in SC (Fig. 8). It is the polar plane of σ, obtained in a completely similar way as in the beginning (Fig. 6). The intersection of this plane with the paraboloid is an ellipse. The different circle transforms can be visualized at a certain level $c = constant$ in the following way: to each point (a, b) is associated the circle σ (a, b, c) for a fixed c $(c = -1, 0, 1)$.

7 Conclusion

This paper has given a new way to look at H.T. using a linear algebra concept which is duality w.r. to a quadratic form.

The different transforms which have been discussed are not simple "avatars" of the H.T. They bring something else. They are "in place" transforms: they do not necessitate a specific representation space. Moreover, two of the circle transforms, $c = -1$ and $c = 1$, can be considered as bounded. Last but not least, their extension to 3D is straightforward which is not the case for H.T.

We develope now software which will be based on the theoretical background that has been presented and will take into account the strong peculiarities of a discrete implementation.

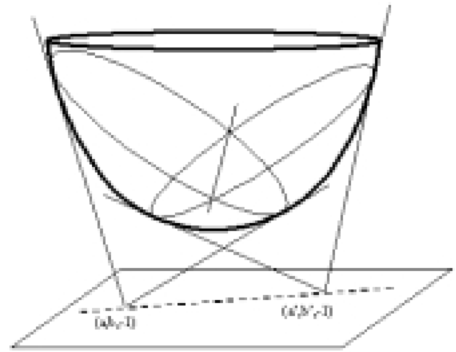

Figure 8: A pencil of circles is represented in SC as a line passing through two particular circles $\sigma(a, b, -1)$ and $\sigma'(a', b', -1)$. The intersection of their polar planes is a line which is the pencil of all circles orthogonal to σ and σ').

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