A new geometrical approach for new Hough-like Transforms

Jean-Marie Becker
Laboratoire Image, Signal, Acoustique (LISA) CNRS EP0092
Ecole Supérieure de Chimie, Physique, Electronique de Lyon (CPE Lyon)
43 Bd du 11 Novembre 1918 (B.P. 2077) 69616 Villeurbanne Cedex (France)
E-mail: becker@cpe.fr

ABSTRACT

Hough Transform, an important tool in image processing, does not use the analytical or geometrical properties of its basic objects, sine curves. Their replacement by other curves, namely circles, has led us to the discovery and the autonomous study of two families of transforms, named Circle and Envelope Transforms. These transforms, internal to the plane of study, are divided into three classes: parabolic (studied in detail), elliptic and hyperbolic, in connection with the Euclidean and the two non-Euclidean geometries. They are shown to be equivalent to Hough Transform. Three "classical geometry" transforms interplay with Envelope Transforms: Reciprocal Polar Transform, Inversion Transform and Pedal Transform. A unified view is brought by the introduction of the "space of circles" equipped with a special quadratic form. This set of transforms can be applied successfully to conic curves in view of their characterization and detection. Almost every concept in this model is generalizable to 3 dimensions in a straightforward manner. Generalization is also promising for gray-level images in the direction of Radon Transform.

Keywords: Image processing, Hough Transform, Geometry, Envelopes, Pedal curves, Conic curves, Inversion, Reciprocal Polar Transform, Quadratic forms.

1. Introduction

Image Processing uses Hough Transform (shortened in "H.T.": see the basic definitions in the Appendix) for the detection of (multiple) alignment. It has been enlarged, often on an "ad hoc" basis, as a tool in curve recognition, especially conic curves retrieval.

H.T. is a point-to-curve transform exchanging alignment and intersection, or, in a practical context, *approximate alignment* in the original (x, y) cartesian coordinates space and *curves clustering* in a specific (θ, p) polar coordinates plane.

H.T. uses *sine curves*. But neither their (rich) analytical properties, nor their (poor) geometrical properties are used.

The purpose of this paper is to exhibit three new types of transforms, each one equivalent to H.T., with sine curves replaced by circles (or by straight lines). These transforms are shown to be strongly connected with Pedal Transform, Inversion Transform and Reciprocal Polar Transform.

The different types can be united in a common theoretical framework, the "space of circles".

The main features of these transforms are studied, together with some "classical geometry" recalls needed to understand them. A more detailed presentation can be found in [1].

Although this paper is theory-oriented, we give a concrete application to a general-purpose curve enhancement algorithm.

Part of the SPIE Conference on Vision Geometry VII • San Diego, California • July 1998
SPIE Vol. 3454 • 0277-786X/98/$10.00

2. Pedal, Circle and Envelope Transforms

We advise to consult the Appendix in order to have an idea about different tools (Pedal transform, Polarity, R.P.T., Inversion) used in this study.

Let us assume that an origin O and an orthonormal basis have been fixed in the Euclidean plane.

Let $M \equiv (a, b)$ a point in this plane.

Let $\begin{cases} \sum_M & \text{be the circle with equation:} \quad x^2 + y^2 - ax - by = 0 \quad \text{(diameter } OM\text{; center in } (a/2, b/2)) \\ \Lambda_M & \text{be the straight line with equation:} \quad ax + by = 1 \end{cases}$

Remarks:

1) Λ_M and $\sum_M$ are exchanged by Inversion I.

2) Λ_M is the polar line of point M.

Let γ be a closed curve, assumed smooth (a C^2 differentiability is convenient).

Definition: The Pedal Transform of γ (denoted $\widehat{\gamma}$) is the locus of the projection of O on the tangent lines to γ.

For a given orientation θ, any straight line $L_{\theta,p}$ which intersects γ belongs to a strip with "support lines" L_{θ,p_1} and L_{θ,p_2} with $p_1 \leq p \leq p_2$.

Property: Let B_γ be the bundle of lines intersecting γ. The associated set $A(B_\gamma)$ (see Appendix) i.e., the region swept by all circles $\sum_M$ when M is the generic point of curve γ, and γ a convex curve, is the interior of its Pedal Transform $\widehat{\gamma}$.

This set will be called the "Circle Transform" of γ.

Even if curve γ is ill-known, for example known by scattered pixels (see Fig. 5), the border of its Circle Transform looks rather precisely outlined. Why is it so? Because this border is more than a simple border; it is the (practical) **envelope** of circles $\sum_M$.

Here is a new notion which, in fact, enlarges the range of Pedal Transform:

We define the **Envelope Transform** (E.T.) of a curve as the envelope $\widehat{\gamma}$ of circles $\sum_M$ when M varies on this curve, when this envelope exists.

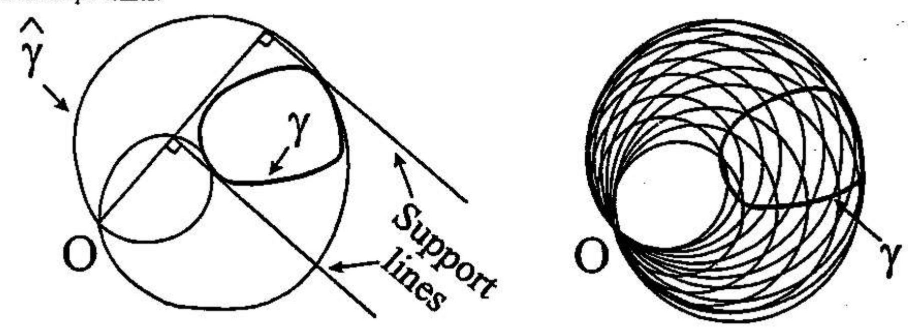

Fig. 1a: A closed convex curve γ and its Pedal Transform (P. T.) $\widehat{\gamma}$.
Fig. 1b: Circles Σ_M (with diameter OM, $M \in \gamma$) "sweep" the interior of the P. T., generating the Circle Transform (C.T.). The envelope of these circles (Envelope Transform) is exactly the. P.T. (see Proposition 1).

3. Envelopes, Inversion and R.P.T.

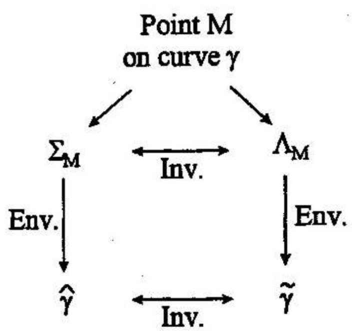

Fig. 2: If M is the generic point of a curve γ, one obtains, by envelope processes, either the Pedal Tr. $\widehat{\gamma}$ (see prop. 1) or the R.P.T. $\widetilde{\gamma}$ of γ (see the definition: indeed Λ_M is the polar line of M); by "transfer", $\widehat{\gamma}$ and $\widetilde{\gamma}$ are, as well, exchanged by Inversion I.

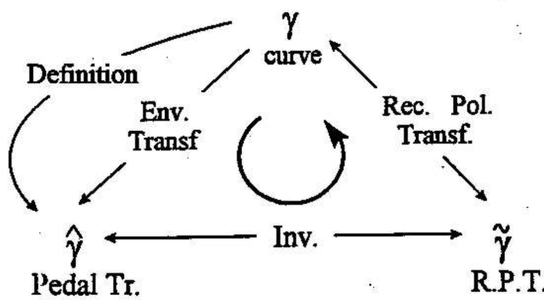

Fig. 3: (linked to Fig. 2 and to prop. 1). From this diagram, an enhancement algorithm (curved arrow) can be built with the following steps (1) Env. Transf. (2) Inversion (3) R.P.T.; a convenient threshold and some dilation steps are needed between steps (1) and (2). See Fig. 5.

Proposition 1: (equivalent to Fig. 3) The Pedal Transform $\widehat{\gamma}$ of a smooth curve γ can be obtained:

- By its definition (locus of the projections of the origin on the tangent lines).
- As Envelope Transform (envelope of circles $\sum_M$, $M \in \gamma$).
- As Inverse Transform of the Reciprocal Polar Transform $\widetilde{\gamma}$ of γ (in closed form: $\widehat{\gamma} = I(\widetilde{\gamma})$).

Proof:

a) Let us give an explicit parametric representation of $\widehat{\gamma}$. We take the notations of Fig. 1.

If $\vec{n_t}$ designates the outward normal *unit* vector in point M_t to curve γ, we have:

$$\overrightarrow{OP_t} = (\overrightarrow{OM_t} \cdot \vec{n_t})\vec{n_t} = \frac{w}{n^2}(y_t', -x_t') \quad \text{with} \quad w \equiv x_t y_t' - x_t' y_t \quad \text{and} \quad n^2 \equiv \left(x_t'\right)^2 + \left(y_t'\right)^2.$$

b) The generic point of a circle envelope is, by definition, the intersection of two "infinitely close circles", i.e., the solution of the following system (where notation $\sum_{M_t}$ is used, in an exceptional way, to designate the left hand side of the corresponding circle's equation):

$$\left\{ \begin{array}{l} \sum_{M_t} = 0 \\ \sum_{M_{t+dt}} = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \sum_{M_t} = 0 \\ \frac{\sum_{M_{t+dt}} - \sum_{M_t}}{dt} = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \sum_{M_t} = 0 \\ \frac{d}{dt}\left(\sum_{M_t}\right) = 0 \end{array} \right.$$

$$\text{Thus:} \qquad (x, y) \in \widehat{\gamma} \text{ for parameter value } t \quad \Leftrightarrow \quad \left\{ \begin{array}{l} x^2 + y^2 - x_t x - y_t y = 0 \\ x_t' x + y_t' y = 0 \end{array} \right. \qquad (1)$$

It means that $(x, y) \in \sum_{M_t} \cap N_t$, where N_t is the line passing through the origin and orthogonal to the tangent to curve γ in point (x_t, y_t). It suffices now to invoke the fundamental property of the diameter of a circle regarding right angles to establish that the solution of system (1) is exactly the generic point of the Pedal Transform.

c) By definition of the Inverse Transform (11), and thanks to the parametric description of the R.P.T. given in the Appendix (10), the generic point of $I(\widetilde{\gamma})$ has the following coordinates:

We will consider only three cases $c = -1, 0, 1$, which, in fact, represent all cases *up to a unit change*. The characteristic properties of circles $\sum_M$ are as follows:

$$\begin{cases} c = 1 & \text{Hyperbolic} & \sum_M \text{ is orthogonal to unit circle.} \\ c = 0 & \text{Parabolic (or Euclidean)} & \sum_M \text{ has } OM \text{ for its diameter (treated in parts 2,3,4)} \\ c = -1 & \text{Elliptic} & \sum_M \text{ intersects unit circle on a diameter of unit circle.} \end{cases}$$

The terms Hyperbolic, Parabolic and Elliptic are classically given for models of plane geometry.

For a fixed c, one can define, as it has been done for the case $c = 0$, a notion of Circle Transform and a notion of Envelope Transform (see Fig. 7a and 7b).

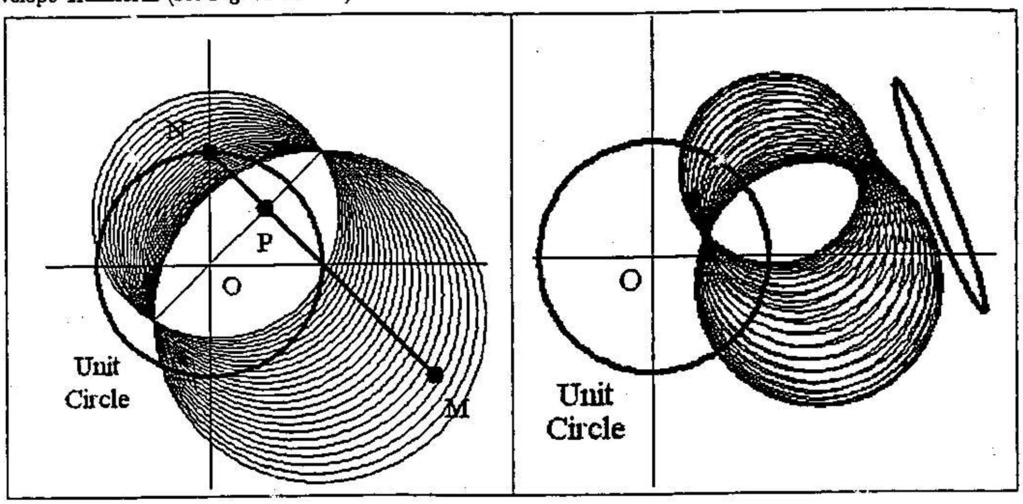

Fig. 7a: Elliptical case ($c = -1$). The Circle Transf. of segment MN. It is still a pencil of circles like in Fig. 6c; but here, circles $\sum_M$ intersect unit circle on a diameter of unit circle.
It can be seen as a "wrapping" of Fig. 6b (interval $(-\pi,\pi]$ is "wrapped" on the unit circle).

Fig. 7b: Hyperbolic case ($c = 1$). The Circle and Envelope Transforms of an ellipse. In this case, all circles $\sum_M$ are orthogonal to the unit circle. As a consequence, the Env. Transf. is globally invariant par standard inversion I. Hence, all the information is in the unit disk: it is bounded.

We will not develope their specific properties, somewhat different from the now known case $c = 0$.

Only one of the most important features, common to all cases, will be treated: the fact that circles can be replaced at any moment by straight lines.

Indeed, let J_c be the point-to-point transform, generalizing inversion I, defined by:

$$X = (x, y) \quad \xrightarrow{J_c} \quad \frac{X}{\|X\|^2 + c} = \left(\frac{x}{x^2 + y^2 + c}, \frac{y}{x^2 + y^2 + c} \right)$$

Remark: If $c < 0$ (elliptic case), a point which belongs to the disk $\|X\| \leq \sqrt{-c}$ has no image by J_c.

Proposition 2: $\quad \forall M : \quad J_c$ exchanges Σ_M and Λ_M.

Proof:

$$(x, y) \in \Sigma_M \quad \Leftrightarrow \quad x^2 + y^2 - ax - by + c = 0 \quad \Leftrightarrow \quad \frac{ax + by}{x^2 + y^2 + c} = 1 \quad \Leftrightarrow \quad J_c(x, y) \in \Delta_M$$

□

Remark: It can be proved that the Circle (resp. Envelope) Transform with parameter $c \neq 0$ generat (resp. curves) that are (globally) invariant by inversion I. Hence, it suffices to know what "happens" in disk; it is why we can call "bounded" the C. Transf. and E. Transf. for $c \neq 0$.

Here is now a global scheme which "encapsulates" the different cases into a single one.

6. The space of circles

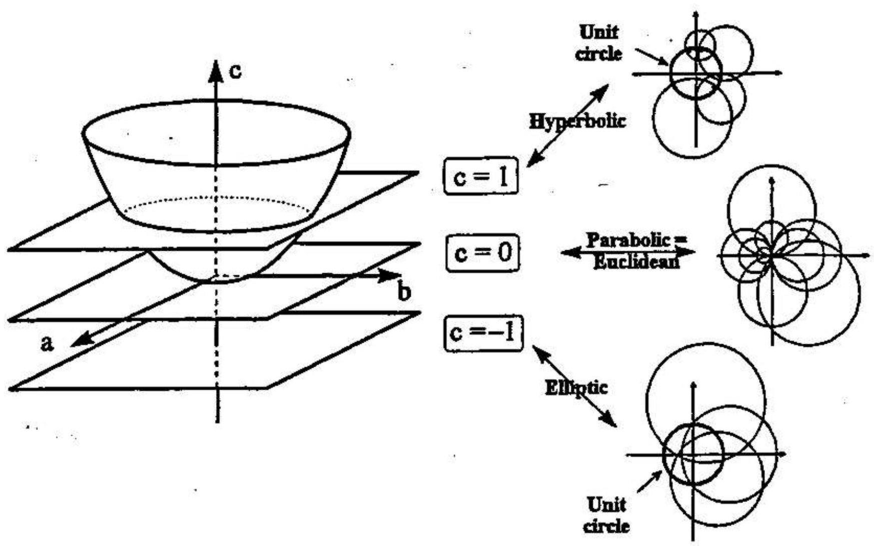

Fig. 8: The space of circles Ω and the three geometries.

The easiest beginning reference for this part is [5], a nice, slightly old-fashioned, little book.

A certain value of c is fixed.

The general equation of circle $\sum_{M,c}$ with center $(a/2, b/2)$ and diameter d is :

$$\begin{cases} \text{with parameters } a, b, c: & x^2 + y^2 - ax - by + c = 0 \\ \text{with parameters } a, b, d: & (x - a/2)^2 + (y - b/2)^2 = (d/2)^2 \end{cases}$$

The different parameters are linked by relationship:

$$4c = a^2 + b^2 - d^2$$

It is parametrization (a, b, c) that will be kept.

The set Ω of (plane) circles appears as a 3D space in which can be placed a rich orthogonality structure.

We will use the term "circle $\sigma = (a, b, c)$" by an identification of the geometrical object and its parameter/coordinate representation.

One can imagine that above and below each point of the plane, a vertical "fiber" exists, and that a (plane) circle with center $(a/2, b/2)$ and coefficient c, instead of being drawn on the plane, is drawn on the horizontal plane at altitude c. For example, at altitude $c = 0$, one finds circles passing through the origin. For positive values of c, no circle can be centered inside the disk with radius $\sqrt{c}$.

It forbids the interior of a certain paraboloid (see remark 2 below).

Let us introduce a "natural" quadratic form on Ω in connection with circles' angles.

Consider Fig. 9. If circles with centers $M = (a/2, b/2)$ and $M' = (a'/2, b'/2)$ and respective diameters d and d' intersect in J with an angle α, we have, considering triangle MJM' (with $\delta =$ distance MM'):

$$4\delta^2 = d^2 + d'^2 - 2dd' \cos(\pi - \alpha) \tag{4}$$

i.e.,
$$(a - a')^2 + (b - b')^2 = d^2 + d'^2 + 2dd' \cos(\alpha)$$

Expanding last relationship and using (3), we obtain:

$$\cos(\alpha) = \frac{2(c + c') - aa' - bb'}{dd'} \tag{5}$$

i.e., by analogy with relationship $\cos(\alpha) = \dfrac{X \cdot X'}{\|X\| \, \|X'\|}$:

$$\cos(\alpha) = \frac{B(\sigma, \sigma')}{\sqrt{Q(\sigma)}\sqrt{Q(\sigma')}} \quad \text{with} \quad \begin{cases} B(\sigma, \sigma') & \equiv \; 2(ct' + c't) - aa' - bb' \\ d^2 = Q(\sigma) & \equiv \; 4ct - a^2 - b^2 \end{cases} \tag{6a}$$

where t, t' confer homogeneity (see [3] about homogeneous coordinates) although we will take here $t = t' = 1$ (points at finite distance).

Remarks:

1) B is a generalized scalar product (bilinear form) and Q a generalized norm (quadratic form).

2) The **isotropic set** (set of vectors σ such that $Q(\sigma) = 0$ is a paraboloid Π defined by

$$4c = a^2 + b^2 \tag{7}$$

$$\begin{cases} \text{Point-circles are on paraboloid } \Pi. \\ \text{Points outside } \Pi \text{ represent (in a unique way) a circle of the plane.} \\ \text{Points inside } \Pi \text{ represent no circle.} \end{cases}$$

Using (5) orthogonality condition $\cos(\alpha) = 0$ for two circles can be expressed as:

$$aa' + bb' = 2(c + c') \tag{8}$$

Remark: Referring to (4), if the second circle is a point-circle ($d' = 0$) with its center on the first circle ($\delta = d/2$), relationship (4) is clearly verified; we will say that

$$\text{circle} \sum \text{passes through point A} \quad \Leftrightarrow \quad \sum \text{is orthogonal to point-circle } \{A\}.$$

In other words, every circle is orthogonal to each of its points, considered as point-circles.

By analogy with the 2D polarity defined in the Appendix, the set of all circles $\sigma \equiv (a, b, c)$ orthogonal to a fix circle $\sigma_0 \equiv (a_0, b_0, c_0)$ constitutes the **polar plane** $P_0 \equiv (a_0, b_0, c_0)^{\perp}$ (see (8)) with equation: $aa_0 + bb_0 - 2(c + c_0) =$ or

$$c = \frac{1}{2}(aa_0 + bb_0) - c_0 \qquad (9)$$

(see Fig. 10)

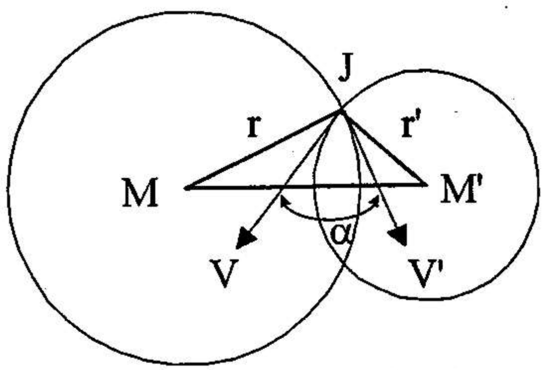

Fig. 9: An angle between curves means the angle between tangent vectors. V and V' at the common point. The angle between JM and JM' is $\pi - \alpha$, because of orthogonality of radius and tangent.

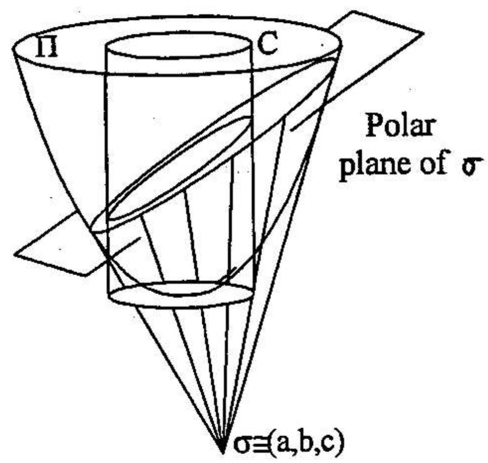

Fig. 10: Polarity in space of circles Ω.

The general connection with Hough Transform is now easy to establish. We fix c_0.

Let $C = [-\pi, \pi) \times \mathbf{R}$ be the vertical unit cylinder built above unit circle $(a = \cos(\theta), b = \sin(\theta))$ (Fig. 10).

Now, consider a point M_0 with cartesian coordinates (a_0, b_0) and polar coordinates $[\theta_0, p_0]$.

The circle $\sum_{M_0} \equiv \sum_{M_0,c} \equiv (a_0, b_0, c_0)$, considered as a point in space Ω, is associated, in a bijective way, to the polar plane of M_0: P_{M_0}.

In this way, circle $\sum_{M_0}$ is associated to the intersection curve of cylinder C with plane P_{M_0} which is an ellipse.

The equation of this ellipse in cylindrical coordinates is, using (9):

$$c = \frac{1}{2}(p_0 \cos(\theta_0) \cos(\theta) + p_0 \sin(\theta_0) \sin(\theta)) - c_0$$

i.e., the equation of the image of circle $\sum_{M_0}$ is: $\qquad c = \frac{1}{2} p_0 \cos(\theta - \theta_0) - c_0$

It suffice now, c_0 being fixed, to take the following affine correspondence between the ordinates in cylinder $C = [-\pi, \pi) \times \mathbf{R}$ and Hough space $[-\pi, \pi) \times \mathbf{R}$:

$$p \xrightarrow{\varphi_{c_0}} c = \frac{1}{2}p - c_0$$

to obtain for the image of circle $\sum_{M_0}$, the sine curve with equation

$$p = p_0 \cos(\theta - \theta_0)$$

associated by H.T. to a point with polar coordinates $[p_0, \theta_0]$; one might think of this operation as an "unrolling" of the cylinder giving Hough plane.

All the process can, of course, be extended to Hough Transforms of shapes (to an area, "swept" by sine curves in a domain, correspond an area "swept" by circles in the other domain...) in a diffeomorphic way.

7. Conclusion and Perspectives

H.T. will not be expelled soon from its important place in image processing!

But the Envelope Transforms that have been introduced can compete seriously with it for certain applications.

These new transforms are more natural, because they do not need a special space of representation; moreover, except for one case $(c = 0)$, they are bounded (with the meaning that all the information is present inside unit disk).

But the most important thing is that this new approach escapes from the "all with sine functions" that looks an intrinsical limitation of H.T.

The "geometric spirit" has provided a breakthrough, with powerful (and relatively simple) tools which are not only theoretical but are adequate on a grid with an evident enhancement of information by the envelope process.

There are many possible research directions with these new tools.

The most evidently important are the gray level extension and the 3D extension.

It is possible to extend H.T. to gray levels in a natural extension.

Almost every concept in this study has a 3D analog. For example the Envelope Transforms generate, in 3D, important surfaces names Dupin cyclides or generalized cyclides which are now quite common in Computer Assisted Design.

8. Appendix[2,3,4]

8.1. The Pedal Transform

The equation of a planar straight line L can be written under the form:

$$x \cos\theta + y \sin\theta = p$$

(θ, p) being the polar coordinates of H, the (orthogonal) projection of O on line L.

Thus, line L is clearly completely determined by point H i.e., by its coordinates (θ, p); this dependance will be reflected into the following notations: $L = L_H = L_{\theta,p}$.

Thus, to a "class" C of straight lines, can be associated the set $A(C)$ of the projections of O on the lines of C, with the evident property $A(\bigcup_{k\in K} C_k) = \bigcup_{k\in K} A(C_k)$

Let S be a set of points; let B_S be the "bundle" of lines passing through (at least a point of) S.

A particular case: the associated set $A(B_M)$ to the bundle of lines B_M ($\equiv B_{\{M\}}$) passing through point M is the circle with diameter OM. We can now return to the general case $A(B_S) = A(\bigcup_{M \in S} B_M) = \bigcup_{M \in S} A(B_M)$, which means that the bundle of lines passing through set S, can be associated to the union of all circles with diameter OM, M being the generic point of set S.

8.2. Polarity

We present here the concept of polarity with respect to the unit circle (this notion is much more general, but we do not need this degree of generality).

Let us consider the correspondence $M \longleftrightarrow \Lambda_M$ which associates to point (a, b) the line Λ_M with equation $ax + by = 1$. The couple M / Λ_M is called the couple pole / polar line. Λ_M is called the **polar line** of point M, and M the **pole** of line Λ_M.

There is an easy-to-prove construction of the polar line when pole M is outside unit circle (see Fig 13a): it suffices to draw the two tangent lines from M to the unit circle and to join the tangency points.

Fundamental property: (Fig. 13 a) If points are aligned on a line L, their polar lines are intersecting in a point M and reciprocally. Moreover, L is the polar line Λ_M of M.

8.3. Reciprocal Polar Transform (R.P.T.)

Alignement des pts → intersec° des polaires

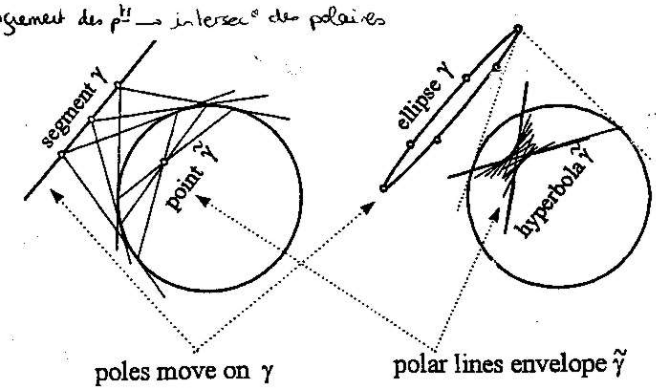

Figure 13: $\tilde{\gamma}$ is the R.P.T. of γ.
On the right: The R.P.T. of a conic curve is a conic curve.
On the left: alignment of poles $\leftrightarrow$ intersection of polar lines;
(a sort of degenerate case of the figure on the right).

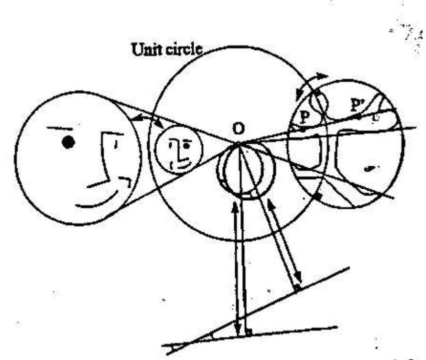

Fig.14: (Standard) Inversion Transform: the different cases.

Let us assume now that, instead of being the generic point of a line segment, M_t is the generic point of a "smooth enough" curve γ. Then straight lines Λ_{M_t} envelope a curve, denoted $\tilde{\gamma}$ (Fig. 13), which is called the Reciprocal Polar Transform (R.P.T.) of γ.

This definition has no symmetry appearance. Nevertheless, we have: