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Explicit formulas for directivity pattern evaluation using Dolph-Chebyshev shading

Raymonde Bel, Jean-Marie Becker and Frédéric Duchemin

L2S2, ICPI Lyon, 31 Place Bellecour, 69288 Lyon Cedex 02, France

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Abstract. — Dolph-Chebyshev shading is a well known method to weigh the response of the elements of an array. Up to now, the weight coefficients could be obtained in two ways: by a Discrete Fourier Transform, or by solving a certain linear system. This paper gives a third method with explicit formulas for the coefficients. We emphasize the theoretical interest of these formulas: for example, their limit conditions lead to the widely used normalized binomial shading coefficients. They are also efficient, as it will be pointed out.

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1. Introduction

A (linear) array is made up of a sequence of individual electro-acoustic transducers called elements. An array can be thought of as a sampled aperture which allows signals to be received or transmitted. We limit our study to the case where the elements are assumed to be punctual.

Shading is a method which improves the reception (or transmission) of signals and permits some degree of control over the directivity pattern of the array. Thus, it implies that the weights (or coefficients) to be applied to the array elements are to be adjusted to provide the most desirable shape for the beam pattern.

One of the main objectives of shading is to get a better precision of the reception zone: to better it, the width of the principal lobe has to be narrowed, but the narrower it is, the higher the secondary lobes are. So a trade-off has to be found. One of the most convenient method, giving the same level for all secondary lobes, is the so-called Dolph-Chebyshev weighting method (Harris, 1978).

More precisely, the Dolph-Chebyshev method for amplitude shading optimizes the far-field beam patterns of arrays by obtaining, for any specified sidelobe level (relative to the level of the main lobe), the narrowest possible main lobe beamwidth, or equivalently, for any specified main lobe beamwidth, the lowest possible sidelobe level with equal maxima (Ziomek, 1985).

However, as pointed out in (Harris, 1978), "the sidelobe structure of the Dolph-Chebyshev window exhibits extreme sensitivity to coefficient errors. This would affect its performance in machines operating with fixed-point arithmetic". Moreover, the shape of the main lobe can also be very sensitive to slight modifications on coefficients. Hence, we may wonder how the weight coefficients can be obtained. By definition, the directivity function of an antenna is the Fourier transform of its aperture function when the spatial frequency is the variable. For representation purposes, the arrival angle computed from the spatial frequency is used (see Figure 1) (Ziomek, 1985).

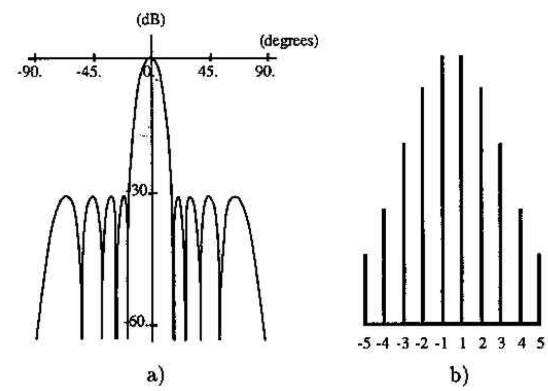


Figure 1. Typical directivity pattern and associate aperture for an even number of elements (see end of Section 3). It should be remarked that numbers from -5 to 5 do not denote abscissae but (symmetrical) indexes of a_n coefficient. a) Directivity pattern. b) Aperture.

Numerically speaking, the most efficient and stable method for weight computing is Fast Fourier Transform (FFT). There is, however, a second way that is the use of an equivalent linear system (LS in short), a method which is recalled in Section 3 of this paper.

Both methods (FFT and LS) are widely used (Helms, 1971; Harris, 1978). They have a common feature: the weights are obtained by an iterative/recursive method.

A thorough analysis of the LS method has led us, after numerous attempts, to explicit formulas for the coefficients.

The main purpose of this paper is to establish these formulas and to obtain binomial shading as a limit of Chebyshev shading. Moreover, we address numerical issues of the different methods.

2. Review of the underlying theory

A short recall of the context is given below with an antenna assumed to be a linear array (see Figure 2 where the notation M is introduced).

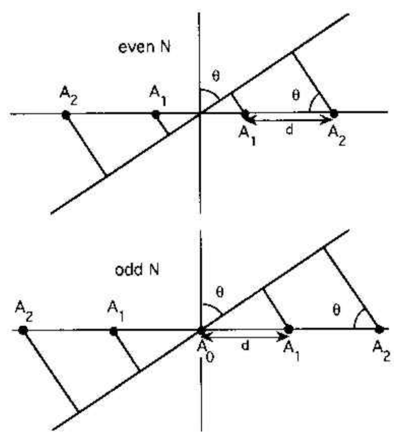


Figure 2. Typical linear array for even $N(N = 2M)$ and odd $N(N = 2M+1)$. Symbol A_n denotes normalized weight coefficients (see Section 3).

A well known analysis of the far-field directivity pattern gives slightly different relations when the number of elements is even $(N = 2M)$ or odd $(N = 2M+1)$:

$$\begin{cases} D(\nu,\theta) = 2\sum_{n=1}^{M} a_n \cos\left[\frac{(2n-1)}{2}\frac{2\pi d\nu}{c}\sin\theta\right] & (1a)\\ \text{if} \quad N = 2M & \\ D(\nu,\theta) = a_0 + 2\sum_{n=1}^{M} a_n \cos\left[n\frac{2\pi d\nu}{c}\sin\theta\right] & (1b)\\ \text{if} \quad N = 2M+1 & \end{cases}$$

where ν is the frequency, θ is the incident arrival angle, d is the inter-element spacing, c is the sound velocity and N is the total number of elements of the array.

Let

$$\phi = \frac{2\pi d\nu}{c}\sin\theta \qquad (2)$$

With Chebyshev polynomials (cf. Appendix B), equations (1a) and (1b) become:

$$\begin{cases} D(\nu,\theta) = 2\sum_{n=1}^{M} a_n \cos\left[(2n-1)\frac{\phi}{2}\right] & \\ \quad = 2\sum_{n=1}^{M} a_n T_{2n-1}(\cos\frac{\phi}{2}) & (3a)\\ \text{if} \quad N = 2M & \\ D(\nu,\theta) = a_0 + 2\sum_{n=1}^{M} a_n \cos\left[2n\frac{\phi}{2}\right] & (3b)\\ \quad = a_0 + 2\sum_{n=1}^{M} a_n T_{2n}(\cos\frac{\phi}{2}) & \\ \text{if} \quad N = 2M+1 & \end{cases}$$

Let $x = \cos\left(\frac{\phi}{2}\right)$:

$$\begin{cases} D(x) = 2\sum_{n=1}^{M} a_n T_{2n-1}(x) \quad \text{if} \quad N = 2M & (4a)\\ D(x) = a_0 + 2\sum_{n=1}^{M} a_n T_{2n}(x) \quad \text{if} \quad N = 2M+1 & (4b) \end{cases}$$

The above equations remain valid whatever shading (i.e. a_n weighting coefficients) are used. We now consider the Dolph-Chebyshev shading, based on a minimality property of Chebyshev polynomials (Dolph, 1946), which gives an optimal compromise between narrow beamwidth and low level sidelobes (Pritchard, 1953; Ziomek, 1985). This shading is defined by the following directivity pattern:

$$D(x) = T_{N-1}(xx_0) \qquad (5)$$

x_0 being such that for $x = 1$ (i.e. $\theta = 0°$):

$$D(1) = T_{N-1}(x_0) = R > 1 \qquad (6)$$

where R is the ratio of the level of the main lobe over the common level of the sidelobes. Using the definition of Chebyshev polynomials (see Appendix B):

$$x_0 = \cosh\left(\frac{1}{N-1}\text{Arg}\cosh(R)\right) \qquad (7)$$

Formulas (4a) and (4b) become:

$$\begin{cases} T_{N-1}(xx_0) = 2\sum_{n=1}^{M} a_n T_{2n-1}(x) \quad \text{if} \quad N = 2M & (8a)\\ T_{N-1}(xx_0) = a_0 + 2\sum_{n=1}^{M} a_n T_{2n}(x) \quad \text{if} \quad N = 2M+1 & (8b) \end{cases}$$

3. The linear system method

We assume here that $N = 2M$ (the case where $N = 2M+1$ being very similar).

Due to its own importance, and also because it will be the basis for the proposition given in Section 4, we first recall the principle of the method.

It can be proved (see Appendix B) that:

$$T_{2j-1}(x) = \sum_{i=1}^{j} t_{ij} x^{2i-1}$$

with

$$t_{ij} = (-1)^{j-i} \frac{2j-1}{2i-1} \binom{i+j-2}{2i-2} 2^{2i-2} \tag{9}$$

Using this formula in equation (8a) and equating coefficients corresponding to equal powers in both members yields the following linear system for the a_n coefficients:

$$\begin{cases} 2a_1 t_{11} + 2a_2 t_{12} + \cdots \cdots + 2a_M t_{1M} = t_{1M} x_0 \\ \qquad 2a_2 t_{22} + 2a_3 t_{23} + \cdots + 2a_M t_{2M} = t_{2M} x_0^3 \\ \qquad \cdots \\ \qquad 2a_M t_{MM} = t_{1M} x_0^{2M-1} \end{cases} \tag{10}$$

System (10) has nonzero diagonal elements; consequently, this upper triangular system has a unique set of solutions a_n.

The linear system method is aimed at the numerical resolution of system (10). For example, when we take $N = 10$, $R = 30$ dB or $R = 31.623$, we get, using (7), $x_0 = 1.108$; hence, using (9) and (10), with all equations divided by 2:

$$\begin{cases} a_1 - 3a_2 + 5a_3 - 7a_4 + 9a_5 = \frac{9x_0}{2} \\ 4a_2 - 20a_3 + 56a_4 - 120a_5 = -\frac{120x_0^3}{2} \\ 16a_3 - 112a_4 + 432a_5 = \frac{432x_0^5}{2} \\ 64a_4 - 576a_5 = -\frac{576x_0^7}{2} \\ 256a_5 = \frac{256x_0^9}{2} \end{cases} \tag{11}$$

Remark: In order to understand how the previous columns are generated, we recall that

$$T_1(x) = x;\ T_3(x) = 4x^3 - 3x;\ T_5(x) = 16x^5 - 20x^3 + 5x.$$

The solutions of system (11) are:

$$a_1 = 4.88 \quad a_2 = 4.29 \quad a_3 = 3.27 \quad a_4 = 2.10 \quad a_5 = 1.26$$

After normalization (i.e. all coefficients divided by a_1):

$$A_1 = 1 \quad A_2 = 0.878 \quad A_3 = 0.669 \quad A_4 = 0.430 \quad A_5 = 0.258$$

These values give the size of the spikes in Figure 1b. The directivity pattern of Figure 1a is obtained with the following physical parameters:

$$c = 1500 \text{ m/s} \quad \nu = 750 \text{ kHz} \quad d = 0.001 \text{ m}$$

4. The explicit formulas for the a_n coefficients

Solving system (10) is an indirect way to obtain the a_n coefficients. Why not "inverting" this system in order to get direct formulas, in the following way:

$$\begin{cases} b_{11}x_0 + b_{12}x_0^3 + \cdots \cdots + b_{1M}x_0^{2M-1} = 2a_1 \\ \qquad b_{22}x_0^3 + b_{23}x_0^5 + \cdots + b_{2M}x_0^{2M-1} = 2a_2 \\ \qquad \cdots \\ \qquad b_{MM}x_0^{2M-1} = 2a_M \end{cases} \tag{12}$$

for certain b_{ij} coefficients ? Apparently, the computation burden for obtaining these coefficients by direct methods (e.g. matrix inversion), looks heavy. However, in an unexpected way, through the study of many special cases, we have found that these b_{ij} coefficients can be expressed by simple formulas, summarized in the following proposition, and proved in Appendix A:

Proposition:
For $i > 0$:

$$a_i = \frac{1}{2} \sum_{j=i}^{M} b_{ij} x_0^{2j-1} \tag{13a}$$

with, for $j \geq i$:

$$b_{ij} = (-1)^{M-j} \frac{N-1}{M+j-1} \binom{M+j-1}{M-j} \binom{2j-1}{j-i} \tag{13b}$$

For $i = 0$ (which arises only when $N = 2M + 1$):

$$a_0 = (-1)^M + \sum_{p=1}^{M} (-1)^{p-1} \frac{N-1}{N-p} \binom{N-p}{p-1} \times \binom{N-2p}{M-p} x_0^{N-2p+1} \tag{13c}$$

5. Binomial shading as a limit of Chebyshev shading

In order to evaluate the directivity pattern, one must normalize the a_n coefficients with respect to a_0 for odd N and with respect to a_1 for even N. We show how to obtain a relationship between the Chebyshev shading coefficients and the binomial ones directly from the normalized parameters.

It is assumed that N is even ($N = 2M$).

Using formulas (13a) and (13b), the normalized coefficients A_n are:

$$A_n = \frac{a_n}{a_1} = \frac{\binom{N-1}{M-n} x_0^{N-1} + \ \ldots}{\binom{N-1}{M-1} x_0^{N-1} + \ \ldots} \tag{14}$$

where dots indicate terms containing smaller powers of x_0.

When R tends towards infinity, so does x_0 (refer to relation (7)). Consequently:

$$\lim_{x_0 \to +\infty} A_n = \frac{\binom{N-1}{M-n}}{\binom{N-1}{M-1}} \tag{15}$$

It means that the A_n Chebyshev coefficients tend to be proportional to the $(N-1)^{\text{th}}$ order binomial coefficients as R tends towards infinity.

Note: After a similar analysis, the odd case gives:

$$A_n = \frac{a_n}{a_0} \quad \text{and} \quad \lim_{x_0 \to +\infty} A_n = \frac{1}{2}\frac{\binom{N-1}{M-n}}{\binom{N-2}{M-1}} \tag{16}$$

This limit condition has already been mentioned (Harrel & Hixson, 1990), but no mathematical proof has yet, to our knowledge, been given. Indeed, Harrel & Hixson (1990) suggest incidentally that the two kind of shadings are inter-related, and that one can be derived from the other, but, once again, without any proof. It should, however, be noted that our approach reduces the allowed freedom degree of the parameters because only R can be chosen.

6. Method comparison and conclusion

Three different methods for the evaluation of the a_n coefficients have been studied:

- (FFT) by applying Inverse Fourier Transform,
- (LS) by solving a linear (triangular) system,
- (F) by using the formulas (13a, b, c).

These different methods have been implemented using C language (more precisely Borland C++) on a PC486. (LS) method has been programmed in a rather straightforward manner, whereas (F) method implementation uses an Horner factorization. (FFT) method uses routines from (Press et al., 1988). The execution time (given in milliseconds) is:

Table 1. *Execution time for the (F) method.*

N	10	20	30	40	50
execution time (ms)	0.2	0.7	1.2	1.9	3.0

Table 2. *Execution time for (FFT) method.*

N	8	16	32	64	128
execution time (ms)	0.3	0.8	2.0	4.1	8.8

(LS) method gives very similar results to (F) method.

These times are, in fact, less important than the precision of the computations, a delicate point that we are going to discuss now.

We have to compare what can be called the "theoretical function" (left hand side of formula (8a)), as opposed to the "reconstructed function" (its right hand side), obtained with the computed coefficients a_n.

Below a certain value of N, the match between these two functions is almost perfect or at least very good; above that value the divergence is very important.

In order to evaluate the discrepancy between these two functions, we have computed the average error for θ between $-20°$ and $20°$. This range has been chosen in order to include the main lobe and a significant number of secondary lobes, when we fix physical parameters like those used in the example of Section 3.

We have worked on two different softwares (C++ and Mathematica); we obtained similar results given below. The divergence threshold for (F) method is for $N = 54$. More precisely:

- for $N < 54$: the discrepancy is $< 2\%$.
- for $N = 54$: it is about 5%.
- For $N = 56$, the divergence is about 30%.

The divergence of (LS) comes much earlier: the 5% error is reached with $N = 34$.

We have chosen to show in Figure 3 the (even) cases $N = 56$ and $N = 36$ where the divergence becomes important (about 30%) for (F) and (LS); on each drawing, the grey curve refers to the theoretical function, and the black one to the reconstructed function. Just above these critical cases, the error is over 100% ...

It must be noted that the two types of divergence exhibited in Figure 3 do not have the same practical consequences. The (F) method gathers the error on the whole diagram and affects the main lobe less than the (LS) method does.

It can be explained by the fact that, for the (LS) method, the first coefficients (a_1, a_2,...), which are known to have a decisive role on the main lobe, accumulate the errors. This is mainly due to successive divisions (pivoting).

A more global reason for the instability of the (LS) method is the ill-conditionning of the system. Let us recall that the "condition number" χ (Golub & Van Loan, 1983) is the ratio of the greatest eigenvalue over the smallest one: as this number increases, the system becomes more unstable. The matrix being triangular, its eigenvalues are the diagonal elements $2t_{ii} = 2^{2i-1}$ (see equation (10)).

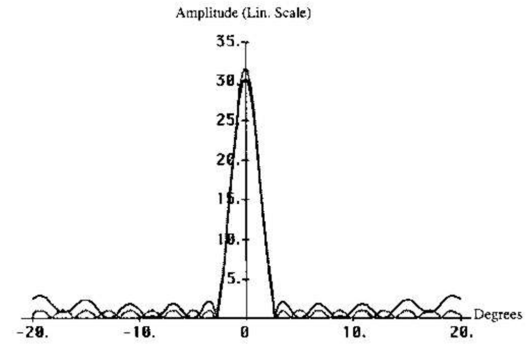
Therefore

$$\chi = \frac{|t_{MM}|}{|t_{11}|} = 2^{2M-1}$$

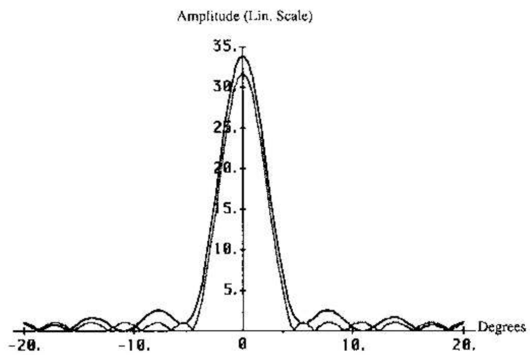
whose growth is exponential.

The (F) method has a larger range of stability. When the divergence phenomenon appears (Figure 3a), due to summation formulas, the error is clearly more dispersed because each coefficient is computed with its own formula, and hence in an independent way.

We also tested the stability of (FFT) method: even for $N > 200$, it is still very stable. It is due to the perfect conditionning of its matrix. But it must be pointed out



a)



b)

Figure 3. a) Divergence start for (F) method ($N = 56$). b) Divergence start for (LS) method ($N = 36$).

that its implementation is less simple than the formula method with Horner scheme.

Remark:

The "theoretical function" should be computed with formulas (3a) and (3b) instead of formula (9).

We now have a new method for the computation of Dolph-Chebyshev coefficients. Its interest, as we have shown, is mainly theoretical; for example it gives the limit binomial distribution. It can be a competitive method for small values of N, because of a better stability with a better dispersion of the error. Therefore it should be preferred to the linear system method, which is proposed in many references.

Acknowledgements

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Appendix A
Proof of the proposition (case $N = 2M$)

We use here the same notations as in Section 3:
Let us recall formulas (9) and (13):

$$t_{ij} = (-1)^{j-i}\frac{2j-1}{2i-1}\binom{i+j-2}{2i-2}2^{2i-2} \qquad \text{(A.1)}$$

$$b_{ij} = (-1)^{M-j}\frac{N-1}{M+j-1}\binom{M+j-1}{M-j}\binom{2j-1}{j-i} \qquad \text{(A.2)}$$

which are valid for $i \leq j$ (for $i > j$, $t_{ij} = b_{ij} = 0$).
Formula (A.1) is proved in Appendix B.
Formula (A.2) will be established in this Appendix.
The system (10) can be written, in matrix notations, as:

$$[t_{ij}] \cdot 2A = D \cdot V \qquad \text{(A.3)}$$

where A and V are the following M-dimensional vectors defined by:

$${}^{t}A = (a_1, a_2 \ldots, a_M) \quad ; \quad {}^{t}V = (x_0,\ x_0^3 \ldots,\ x_0^{2M-1})$$

where letter t refers to the transposition operator and D is the diagonal matrix whose (diagonal) entries are the t_{iM}.
System (12) can be written as:

$$[b_{ij}] \cdot V = 2A \qquad \text{(A.4)}$$

Combining (A.4) and (A.3) gives:

$$[t_{ij}] \cdot [b_{ij}] \cdot V = D \cdot V \quad \text{(for any } V\text{)}$$

It means that we have now to establish the following matrix identity:

$$[t_{ij}] \cdot [b_{ij}] = \begin{pmatrix} t_{1M} & & & \\ & t_{2M} & & \\ & & \ldots & \\ & & & t_{MM} \end{pmatrix} \qquad \text{(A.5)}$$

Let c_{ij} be the general coefficient of the product matrix:

$$c_{ij} = \sum_{u=1}^{M} t_{iu}b_{uj} = \sum_{u=i}^{j} t_{iu}b_{uj} \qquad \text{(A.6)}$$

Using (A.1) and (A.2) in (A.6) gives:

$$c_{ij} = \sum_{u=i}^{j}(-1)^{M+i-j+u}\frac{2u-1}{2i-1}\frac{2M-1}{M+j-1} \times \binom{i+u-2}{2i-2}\binom{M+j-1}{M-j}\binom{2j-1}{j-u}2^{2i-2} \qquad \text{(A.7)}$$

Hence, (A.5) will be proved when the following equalities are established:

$$\begin{cases} c_{ij} = t_{iM} & \text{whenever} \quad i = j \\ c_{ij} = 0 & \text{whenever} \quad i \neq j \end{cases} \qquad \text{(A.8)}$$

The first case ($i = j$) is straightforward using equations (A.7) and (A.1) :

$$c_{ii} = (-1)^{M+i}\frac{2M-1}{M+i-1}\binom{M+i-1}{M-i}2^{2i-2}$$
$$= (-1)^{M-i}\frac{2M-1}{2i-1}\binom{M+i-2}{2i-2}2^{2i-2} = t_{iM}$$

We now assume $i \neq j$. What needs to be shown is that $c_{ij} = 0$, that is to say, using (A.7) with the suppression of unimportant factors:

$$\sum_{u=i}^{j}(-1)^u(2u-1)\binom{i+u-2}{2i-2}\binom{2j-1}{j-u} \stackrel{?}{=} 0 \qquad \text{(A.9)}$$

Replacing parameters i, j and u by

$$n = u - i, \qquad q = j - i \quad \text{and} \quad k = 2i - 2,$$

we have to show that the following sum is null (for any $k \geq 0$ and $q \geq 1$):

$$L = \sum_{n=0}^{q}(-1)^n(2n+k+1)\binom{k+n}{k}\binom{k+2q+1}{q-n} \qquad \text{(A.10)}$$

In order to prove that $L = 0$, we introduce the following generating function:

$$f(u,x) = \frac{1}{u^q}\frac{1}{(1-ux)^{k+1}}(u-x)^{k+2q+1} \qquad \text{(A.11)}$$

in which we will later find L as a certain coefficient.

The formal expansion

$$f(u,x) = u^{-q}\sum_{n=0}^{+\infty}\binom{k+n}{k}(ux)^n$$
$$\times \sum_{p=0}^{k+2q+1}(-1)^p\binom{k+2q+1}{p}u^{k+2q+1-p}x^p \qquad \text{(A.12)}$$

can be considered as a series in x whose coefficients are functions of variable u:

$$f(u,x) = \sum_{r=0}^{+\infty}c_r(u)x^r$$

Let us now consider the q-th coefficient, whose development, obtained by the constraint $n + p = q$ in (A.12), is:

$$c_q(u) = \sum_{n=0}^{q}(-1)^{q-n}\binom{k+n}{k}\binom{k+2q+1}{q-n}u^{2n+k+1} \qquad \text{(A.13)}$$

(A.13) yields, by derivation and evaluation for $u = 1$ (see (A.10)):

$$\frac{\mathrm{d}c_q}{\mathrm{d}u}(1) = (-1)^q L \qquad \text{(A.14)}$$

Thus, the question is now to show that $\frac{\partial c_q}{\partial u}(1) = 0$. $\frac{\partial c_q}{\partial u}$ (1) is the coefficient of x^q in $\frac{\partial f}{\partial u}|_{u=1}$ (cf. (A.13)), which is easy to obtain by a (logarithmic) derivation of (A.11):

$$\frac{\partial f}{\partial u} = \left[-\frac{q}{u} + (k+1)\frac{x}{1-ux} + \frac{k+2q+1}{u-x}\right]f$$

Whence

$$\frac{\partial f}{\partial u}|_{u=1} = (1+k+q)(1+x)(1-x)^{2q-1}$$

$$= \pm(1+k+q)(1+x)\left[\ldots\binom{2q-1}{q-1}x^{q-1} - \binom{2q-1}{q}x^q\ldots\right]$$

whose degree q coefficient is

$$\pm(1+k+q)\left[\binom{2q-1}{q-1} - \binom{2q-1}{q}\right]$$

i.e. zero, as desired.

Remarks:

1) Formula $\frac{1}{(1-X)^{k+1}} = \sum_{n=0}^{+\infty}\binom{k+n}{n}X^n$ can be obtained by derivating k times the geometric series formula: $\frac{1}{1-X} = \sum_{m=0}^{+\infty}X^m$.
2) Another proof may be given using Dixon's formula for hypergeometric functions (Graham et al., 1989).
3) The change of variable (A.9) implies that k is even; we did not use that restriction, i.e. we have proved a slightly more general result.
4) We do not give a proof of the proposition for the case $N = 2M + 1$ because it is very similar to the even case.

Appendix B
Proof of equation (9)

There are two ways to define the n^{th} order Chebyshev polynomial (Abramowitz & Stegun, 1965):
a) With circular or hyperbolic cosine:

Case $x \geq 0$:

$$T_n(x) = \begin{cases} T_n(\cos(\alpha)) = \cos(n\alpha) & \text{for} \quad 0 \leq x = \cos(\alpha) \leq 1 \\ T_n(\cosh(a)) = \cosh(na) & \text{for} \quad x = \cosh(a) \geq 1 \end{cases}$$

Case $x < 0$:

$$T_n(x) = (-1)^n T_n(-x)$$

b) As a polynomial:

$$T_n(x) = \frac{1}{2}\sum_{k=0}^{\xi}(-1)^k\frac{n}{n-k}\binom{n-k}{k}(2x)^{n-2k}$$

where ξ is the integer part of $n/2$.

In the odd case $n = 2j - 1$, the previous formula can be written as:

$$T_{2j-1}(x) = = \frac{1}{2}\sum_{k=0}^{j-1}(-1)^k \frac{2j-1}{2j-1-k}\binom{2j-1-k}{k}(2x)^{2j-1-2k}$$

With the index change $j-k=i$:

$$T_{2j-1}(x) = \frac{1}{2}\sum_{i=1}^{j}(-1)^{j-i}\frac{2j-1}{i+j-1}\binom{i+j-1}{j-i}(2x)^{2i-1}$$

Hence

$$T_{2j-1}(x) = \frac{1}{2}\sum_{i=1}^{j}(-1)^{j-i}\frac{2j-1}{2i-1}\binom{i+j-2}{2i-2}(2x)^{2i-1}$$

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