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A MONTE CARLO METHOD FOR MEASUREMENT OF STRIP WIDTHS. AN APPLICATION IN BONE HISTOMORPHOMETRY

Jean-Marie Becker, Daniel Chappard*, Christian Alexandre*

Laboratoire TSI: Traitement du Signal et Instrumentation UA CNRS 842
et *LBTO: Laboratoire de Biologie du Tissu Osseux.
Université de St Etienne, 42023 St Etienne Cedex (France)

ABSTRACT

The measurement of strip widths on a digitized image can be done with a Monte Carlo method using random particles whose paths follow a staircase pattern inside the strip, in one of the four quadrants built around their "birthplace".

The main purpose of this note is to derive the formula

$$E(L) = D \,.\, \sqrt{2} / \cos(\Theta - \pi/4)$$

linking the mathematical expectation of the walk length of a particle born in O, the local inclination Θ of the strip and the distance from O to the boundary of the strip in a given quadrant.

Keywords : bone histomorphometry, Monte Carlo method, random walk, width measurement.

INTRODUCTION

Bone cells synthetise layers of uncalcified matrix (the osteoid tissue). The mineralization process, which occurs in a second step, may be impaired in several pathological disorders, associated with a drastic increase of osteoid seams thickness. The knowledge of the thickness of these strips allows a differentiation between different kinds of metabolic bone diseases. Indirect approximation of this thickness can be derived from volume V and surface S, but it has been shown to overestimate mean thickness provided by direct control measurements with calibrated microscopic eyepieces (Vedi and Compston, 1984).

This thickness can satisfactorily be obtained by the method of <u>random linear intercepts</u> (Weibel, 1980). The purpose of this note is to show that, on an image analyser, this can also be done with <u>random moving points</u>, with simple theoretical tools which will be discussed below.

We shall be interested in the distribution of the (apparent = two dimentional) width of a strip of osteoid on profiles, from which the effective thickness can easily be obtained.

The strips will be considered, after a convenient thresholding, as made out of black points on a white surrounding. An origin point O being chosen, somewhere inside the strip, we have to estimate the "width in O" The profile's boundary will be assumed to be a line, at least locally.

Point O will be taken as the birthplace of several "particles" allowed to move randomly North, South, East or West as long as they stay in the black area. When they reach a white point, they "die".

In short, the basic idea is "the longer the life, the larger the strip".

This fuzzy formulation has to be made more precise.

A pure random walk would yield a very unefficient computing method.

An evident improvement is to consider the four quadrants around O, and, in each one, to give birth to particles compelled to move in two directions only. For example, in North East quadrant, particles are bound to go North or East, with equal probability. In this way, every move increases the distance from O to the particle. Fortunately also, the problem becomes in this way reasonnably tractable, as we are going to see it.

First, we have to master the situation in a single quadrant. All our explanations will be given in North East quadrant.

We introduce here two notations used throughout the text :

L = length of the path of a particle (called also "length of life").

D = distance from O to the boundary of the strip situated in the quadrant of interest .

Let us first tackle two particular cases worth of interest. The first with inclination $\Theta = \pi/4$ of the strip, the other with $\Theta = \pi/2$. It will be a valuable help for a better understanding of the relation between the mathematical expectation E(L) of the lengthes and the distance D from O to the boundary of the strip.

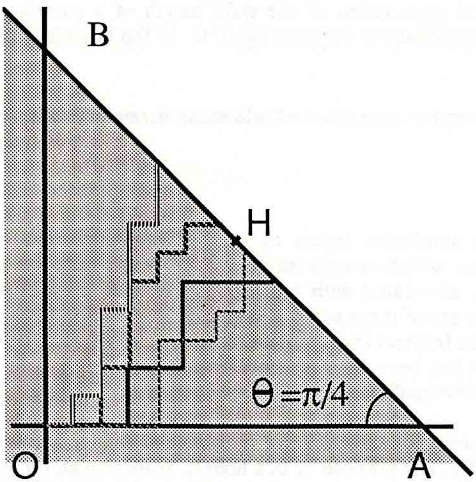

Fig. 1.: particular case $\Theta = \pi/4$

PARTICULAR CASE $\Theta = \pi/4$ (fig. 1) :

Fig. 1 may be seen as a Galton's board: in this way, it is evident that the particles' places of death are distributed according to binomial law B(n,1/2).

Any particle leaving O has exactly the same path length as any other, and this common length is nothing but length OA.

So, the relation

$$OA = \sqrt{2} \, . \, OH \qquad \text{may be written}$$

$$E(L) = \sqrt{2} \, . \, D \qquad \text{whose form is}$$

$$\boxed{E(L) = k \, . \, D} \qquad (1)$$

PARTICULAR CASE $\Theta = \pi/2$ (fig. 2) :

The shortest distance $D = OH$ from O to the boundary is equal to the number of "narrow" vertical strips.

The random variable L can be written as

$$L = L_1 + L_2 + ... + L_D$$

where L_i is the length of the fragment of the path situated into the strip :

$$i - 1 \leq x < i$$

As all random variables L_i follow the same law defined by

$$P(L_i = k) = 1/2^k \qquad (k = 1, 2...)$$

and whose expectation is 2, L follows a negative binomial law (Feller, 1968) and its expectation is :

$$E(L) = 2 \, . \, D$$

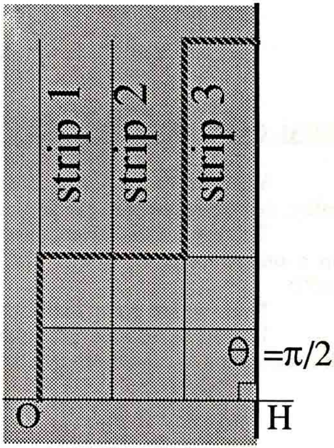

Fig. 2.: case $O = \pi/2$

Once more, we find a relation

$$E(L) = k \, . \, D$$

We are going to show below why formula (1) stands in the general case.

Furthermore, we shall prove that the expression of k as a function of Θ is

$$\boxed{k = \frac{\sqrt{2}}{\cos(\Theta - \pi/4)}} \qquad (2)$$

Now, before the general case, we check that E(L) is linearly related to D.

THE LINEAR DEPENDANCY E(L) = k.D

Let us show that, when the distance is doubled, so is the length expectation (see fig. 3). From there, a classical reasoning will establish the linearity.

We may write $L = L' + L''$

where L' and L" stand, respectively, for the lengths of the paths before and after the particle has crossed AB.

L" follows the same law as L' because, wherever AB is crossed, all the path in the second zone is located inside a triangle identical to triangle OAB. Taking the expectations:

$$E(L) = E(L') + E(L'') = 2 \,.\, E(L').$$

GENERAL CASE : FORMULA k = k(Θ) :

We refer to fig. 4. By consideration of symetries, we may assume, without loss in generality, that the range of angle Θ is between $\pi/4$ and $\pi/2$.

The key idea is the introduction of line AB (inclination $\pi/4$). Every particle crosses AB in a unique point E_i (i,n-i). The distribution of these points along AB is binomial (B(n,1/2)).

Now, we are going to obtain k through a discrete equivalent of an integral equation.

L can be splitted into two random variables: the length of the path before AB, which is a constant (n) and the length L' of the path after AB:

$$L = n + L'$$

Taking expectations : $k.D = n + E(L')$

and then, according to the binomial weighting :

$$k.n.\sin\Theta = n + \sum_i^n (1/2)^n \binom{n}{i} E(L_i)$$

(L_i being the length of the path of a particle "born" in E_i and which "dies" when reaching AA'). The similarity of triangles E_iFG and OAA' gives the relation:

$$E(L_i) = k \,.\, D_i \qquad (D_i = \text{distance from } E_i \text{ to AA'})$$

from which $E(L_i) = k \,.\, E_iA \,.\, \sin(\Theta - \pi/4)$

or $E(L_i) = k \,.\, i.\sqrt{2} \,.\, (\sin\Theta - \cos\Theta)/\sqrt{2}$

Then, formula (3) becomes :

$$k.n.\sin\Theta = n + k \,.\, (\sin\Theta - \cos\Theta) \,.\, \sum_i^n i \; (1/2)^n . \binom{n}{i}$$

The last summation being the expectation n/2 of binomial law B(n,1/2),

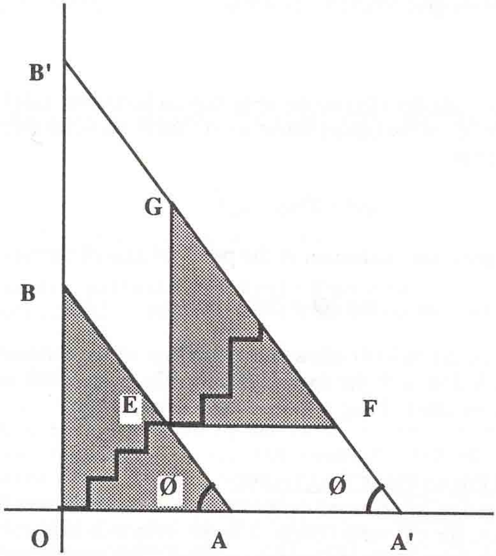

Fig. 3.: A line which is twice as distant gives rise to a doubling of E(L)

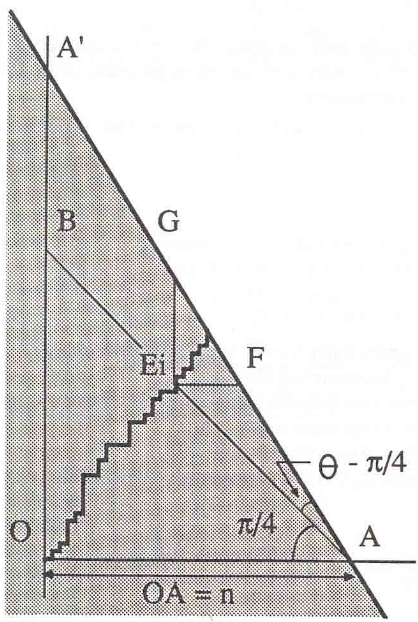

Fig. 4.: General case

staightforward computations give formula (2) for k.

THE LOCAL WIDTH:

The angle Θ made (locally) by the strip with an horizontal axis is easily obtained if we keep track of the "cloud" of the places where the different particles die. The knowledge of the quadratic form of inertia

$$ax^2 + 2bxy + cy^2$$

associated to the cloud gives the inclination of the principal axis of inertia of this cloud by the formula

$$\tan 2\Theta = 2b/(a - c) \qquad (4)$$

Formulas (1), (2) and (4) allow the estimation of the different distances D_{NE}, D_{NW}, D_{SE} and D_{SW} linked with the four quadrants. The local width will be taken as the smallest among the two numbers $D_{NE} + D_{SW}$ and $D_{NW} + D_{SE}$.

FROM LOCAL TO GLOBAL CALCULATIONS :

Points O need to be scattered uniformly on all the black area, but , in this way, it must be understood that, for the same length, a larger strip will have more points O than a smaller one. In this way, large strips are overrepresented. This trouble may be overcomed by changing the contents c_i of the histogram of the widths by c_i / i.

The general efficiency and accuracy of this method on a classical computer is good. But this efficiency could be enhanced by using parallel computing with a convenient random numbers (random bits) generator.

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