Making Connections: A Graphical Construction

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Suppose you have a list of club members who would like to exchange letters with one or more other members. The most active member tells you he wants 12 pen pals, the next two members want 11 pen pals each, and so on. Each member requests a specified number of pen pals. When you check up, you find that you have 74 letters to send out, and they've come to you from the 16 people on your list in the following batches: 12, 11, 11, 5, 5, 5, 5, 5, 4, 4, 2, 1, 1, 1, 1, 1.

In terms of number theory this sequence represents a **partition** of the integer 74, a collection of positive integers adding up to 74. (See [**2**] or [**3**].) Your problem is to make sure that each person receives as many letters as he's sent out, that is, to find an appropriate **graph**, a set of points called **vertices** (representing the members) with connecting lines called **edges** between some of them (representing a unique exchange of letters between a pair).

The **degree** of a vertex of a graph is the number of edges extending from it. A list consisting of the degrees of each vertex of a graph is called a **degree sequence**. Not all sequences of integers can be degree sequences of graphs. For example, there is no graph whose vertices have degrees 4, 2, 2, 2. (You might try to draw such a graph; you will find that there will be only three vertices for the four edges from the first vertex to meet.) Nor will 4, 2, 2, 2, 1 do (since each edge is counted twice, the sum of the numbers in a degree sequence must be even). Thus, given the original problem, you might not be able to find a graph with a degree sequence that solves it.

The question of whether a given partition is the degree sequence of a graph is answered by a well-known theorem of Erdős and Gallai [**1**]. The Erdős-Gallai condition gives the criterion in combinatorial terms that (roughly) ensure that early numbers in the partition are not too large in relation to later numbers. In this note we offer another criterion; it accomplishes the same thing in number-theoretic terms. We shall illustrate how to use our criterion by constructing a graph with the partition of 74 in our problem as its degree sequence.

With each partition $(d_1, d_2, \ldots, d_q)$ with $d_i \geq d_j > 0$ for $i < j$, we can associate an array of dots called its **Ferrers diagram** [3]. This is a set of dots aligned in rows and columns, both in nonincreasing order, so that the ith row has d_i dots and all rows begin with a dot in the first column. If we let a dot represent a letter sent out, we'll get the Ferrers diagram F for the original problem, shown in FIGURE 1.

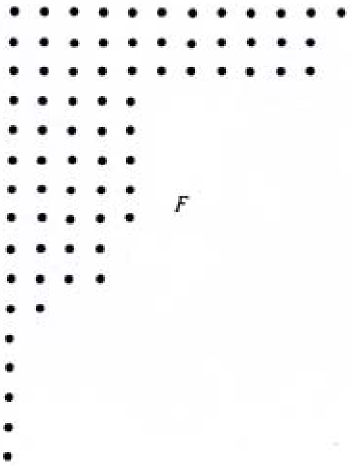

FIGURE 1

Counting the number of dots in each column produces another partition of 74, called the **conjugate partition**: 16, 11, 10, 10, 8, 3, 3, 3, 3, 3, 3, 1. As a first attempt to assign pen pals, you might label the columns with names of members in the same order as the rows, then send each letter out to the person whose name heads the column in which it is represented. The column totals will tell you how many letters each member receives. But that won't do. No member wants to receive a letter from himself, so you'll certainly have to remove dots from the diagonal that starts at the top left. You'll have to change the column totals—but not those of the rows—to make sure that each person receives as many letters as he's sent. The result, with a minor change of notation, will be the **adjacency matrix** of a graph, a symmetric matrix of zeros and ones, with $a_{i,i} = 0$, $a_{i,j} = 0$ if vertices p_i and p_j are not connected by edges, and $a_{i,j} = 1$ if p_i and p_j are connected by edges.

We shall construct a transformation from Ferrers diagram to adjacency matrix. In doing this we'll prove the following theorem, which contains our criterion for a partition to be the degree sequence of a graph.

THEOREM. *Let D be a partition of an even integer $2n$, $D = (d_1, \ldots, d_q)$ with $d_1 \geq \cdots \geq d_q > 0$ having conjugate partition $K = (k_1, \ldots, k_r)$, $k_1 \geq \cdots \geq k_r > 0$. Let h be the largest integer such that $d_h \geq h$. D is the degree sequence of a graph if and only if for each r such that $1 \leq r \leq h$ the following inequality holds*:

$$\sum_1^r d_i \leq \sum_1^r (k_i - 1). \tag{1}$$

The corner square of h^2 dots in the Ferrers diagram is called the **Durfee square** [2]; it is the largest subsquare of dots in the Ferrers diagram. Note that $d_{h+1} < h+1$, $k_h \geq h$, and $k_{h+1} < h+1$. In FIGURE 1, $h = 5$.

The above theorem can be proved using the Erdős-Gallai theorem [4]; in fact the two theorems are equivalent. The Erdős-Gallai condition is that for all r, the following inequality holds:

$$\sum_1^r d_i \leq r(r-1) + \sum_{r+1}^q \min\{r, d_i\}.$$

Our proof is a constructive one; our theorem could provide an alternative proof of the Erdős-Gallai criterion.

Proof. It is easy to prove that condition (1) is necessary. For suppose there exists a graph having D as the degree sequence. The vertices with greatest degree will have as labels the smallest numbers. The adjacency matrix A of the graph has d_i as the sum both of its ith row and of its ith column. (In either case, this is the number of edges incident with the vertex p_i.) We can construct a new matrix B from A (B will look like a Ferrers diagram if we identify the 1's with dots and 0's with blanks) as follows. To form the ith row of B, shift the positions of the 1's and 0's in the ith row of A so that all of the 1's are now placed in the first d_i columns and the 0's fill out the rest of the row. Clearly B has the same row sums as A, but the column sums are almost surely different. To examine the column sums, we suppose that the transformation from A to B is carried out in two stages. First, note that for $i \leq h$ it is true that $d_i \geq d_h \geq h$, so there must be a 1 at $a_{i,h}$ or to its right, and this 1 can be moved left to $a_{i,i}$. If we let c_i be the ith column sum of the matrix after this shift for all $i \leq h$, then for such i, we have $c_i \geq d_i + 1$ and so for $r \leq h$,

$$\sum_1^r (d_i + 1) \leq \sum_1^r c_i. \tag{2}$$

After this first shift, we move all 1's in each row as far as possible to the left to obtain the matrix B. If we let k_i be the ith column sum of B, then for all r

$$\sum_1^r c_i \leq \sum_1^r k_i. \tag{3}$$

Using (2) and (3), we see that for $r \leq h$

$$\sum_1^r d_i \leq \sum_1^r (k_i - 1),$$

so that (changing 1's to dots) B provides a Ferrers diagram with the required property.

To prove the sufficiency of (1) we will transform a Ferrers diagram F to an adjacency matrix, and this requires a bit more work. Our problem is to move dots to the right so that the number of dots in each row remains the same and columns become symmetric with rows. One way to do this would be to use the Havel-Hakimi process ([1], Chap. 6). The construction below is better adapted to make use of our condition (1). It is a process by which we construct one row and column of an adjacency matrix at a time, with the resulting adjacency matrix corresponding to a graph which has D as its degree sequence. There are two cases, requiring two different strategies. First, if inequality (1) becomes an equality for any $r \leq h$, we work with the diagram F to eliminate row r. (If there is more than one such r, choose the largest.) Second, if inequalities (1) are all strict, we begin with row h, the last row such that $d_h \geq h$.

The pen pal problem and its Ferrers diagram in FIGURE 1 were designed to satisfy condition (1) in order to illustrate the various situations that might arise in each case. In this example $h = 5$ and $q = 16$. For convenience we shall refer to the intersection of the ith row and jth column of the Ferrers diagram F as position $a_{i,j}$. In altering F, we shall always *move dots to the right without changing the number of dots in any row*. After altering the placement of dots in a given row of F and if necessary, increasing the number of dots in the corresponding column, we then substitute 1's for dots and 0's for blanks to fill in a 16×16 adjacency matrix.

In FIGURE 1, the only $r \leq h$ for which the two sides of (1) are equal is $r = 3$. The left side becomes $\Sigma_1^3 d_i = 34$ and the right side is $\Sigma_1^3 (k_i - 1) = 37 - 3 = 34$. To fill in row 3 and column 3 of the adjacency matrix, we first empty the diagonal position of F: the dot at $a_{3,3}$ is moved to the first empty position on the right end of the row, which is $a_{3,12}$. Row 3 has 11 dots and column 3 now has 9 dots, so we must increase the number of dots in column 3 by 2. To do this, we will

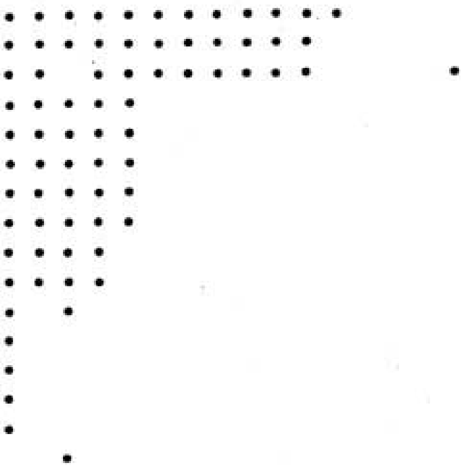

FIGURE 2

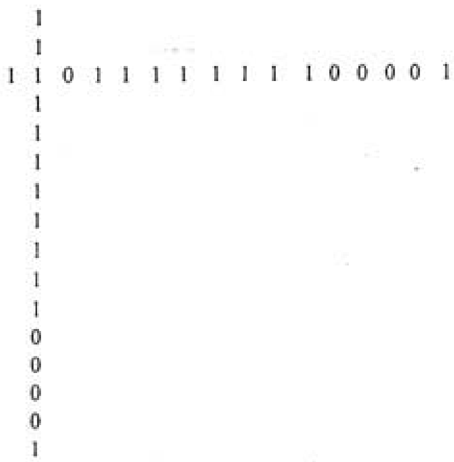

FIGURE 3

move rightward to column 3 two dots chosen from below the tenth row in columns to the left of column 3. Although inspection of the diagram in FIGURE 1 shows that dots in column 1 and 2 below row 10 are available to be moved (right) to column 3, in the general case, it is condition (1) that guarantees the existence of these dots, available to be moved in a similar manner, to create a symmetric row and column of dots. In our case, we first move one dot from $a_{11,2}$ to $a_{11,3}$. Since no other dots occur in column 2 below row 11, we look to the next column to the left (column 1) and move a dot in any row below row 11. We arbitrarily choose the lowest and shift a dot from $a_{16,1}$ to $a_{16,3}$. Finally we make row 3 symmetric with column 3; we move dot $a_{3,12}$ to $a_{3,16}$. The new configuration of dots we have obtained from F is shown in FIGURE 2. We use this configuration both to obtain row 3 and column 3 of the adjacency matrix we seek to construct and to produce a new Ferrers diagram to continue the construction. Replacing dots by 1's and blanks by 0's in row 3 and column 3 of FIGURE 2 results in the corresponding row and column of the adjacency matrix under construction (see FIGURE 3). Striking out row 3 and column 3 in the configuration of FIGURE 2 produces the configuration in FIGURE 4(a); from this we can obtain a new Ferrers diagram F^* shown in FIGURE 4(b) (just eliminate spaces so as to abide by Ferrers diagram rules). F^* is the Ferrers diagram of a degree sequence D^* of length $q^* < q$ (in our example, $q^* = 14 = q-2$), and the size h^* of the Durfee square in F^*, is $h-1$. (This is because the dots eliminated from F were from row 3 and column 3.) If we let K^* be the conjugate partition of D^*, then it is an easy exercise to show that F^* satisfies the conditions of the theorem for $1 \leq r \leq h^*$.

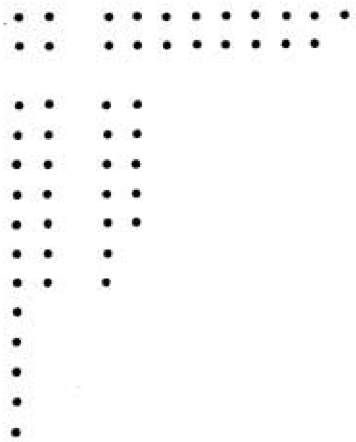

FIGURE 4 (a)

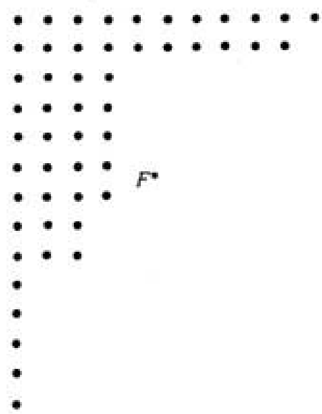

FIGURE 4 (b)

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0 1 1 1 1 1 1 1 1 1 0 1 1 1 0 0
1 0 1 1 1 1 1 1 1 1 1 0 0 0 1 0
1 1 0 1 1 1 1 1 1 1 1 0 0 0 0 1
1 1 1
1 1 1
1 1 1
1 1 1
1 1 1
1 1 1
1 1 1
1 1 1
0 1 1
1 0 0
1 0 0
1 0 0
0 1 0
0 0 1
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FIGURE 5

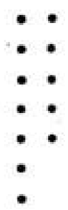

FIGURE 6

The process is repeated with F^*. In every case, if we start with row r in F for which the sums in (1) are equal, equality will hold for sums of $r-1$ terms in F^*, and so it is in our example. For $r=2$ in our new diagram, $d_1^* + d_2^* = 21$ and $k_1^* + k_2^* = 23$. We work on row 2 and column 2 as before, then repeat the process with row 1 and column 1. We have progressed in matrix building to FIGURE 5.

At this stage we have struck out all dots in the first three rows and columns of our original Ferrers diagram F, and all that remains is shown in FIGURE 6. This diagram is to be transformed into a 13×13 adjacency matrix, filling in the southeast corner of FIGURE 5.

The Ferrers diagram in FIGURE 6 gives us a chance to show the strategy of our construction when all inequalities (1) of the theorem are strict. In this case we must work on row h, the row with the largest subscript such that $d_h \geq h$. If we look at it as a fresh problem, our diagram in FIGURE 6 has $h=2$, $d_1 = d_2 = 2$, $k_1 = 7$, and $k_2 = 5$. So (1) is a strict inequality for $r=1$ and $r=2$. Again we need $a_{i,i} = 0$, and so dot $a_{2,2}$ of our example is moved to position $a_{2,3}$. If $k_2 < d_2$ we proceed as in the first case and move dots rightward to column 2, then adjust the position of dots in row 2 to make row 2 and column 2 symmetric. Since $k_2 > d_2$, we instead retain $k_2 = 2$ dots at the bottom of column 2 (i.e., dots $a_{5,2}$ and $a_{4,2}$ retain their positions) and move all dots in column 2 above these one position to the right. Now to make row 2 symmetric with column 2, we move dot $a_{2,3}$ to position $a_{2,5}$ and dot $a_{2,1}$ to position $a_{2,4}$. The adjusted diagram is shown in FIGURE 7; we use this in the same way that the configuration in FIGURE 2 was used earlier.

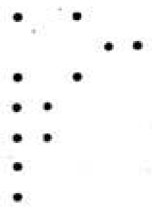

FIGURE 7

```
0 1 1 1 1 1 1 1 1 1 0 1 1 1 0 0
1 0 1 1 1 1 1 1 1 1 1 0 0 0 1 0
1 1 0 1 1 1 1 1 1 1 1 0 0 0 0 1
1 1 1   0
1 1 1 0 0 0 1 1 0 0 0 0 0 0 0 0
1 1 1   0
1 1 1   1
1 1 1   1
1 1 1   0
1 1 1   0
0 1 1   0
1 0 0   0
1 0 0   0
1 0 0   0
0 1 0   0
0 0 1   0
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FIGURE 8